QUASI-RIGIDITY OF HYPERBOLIC 3-MANIFOLDS AND SCATTERING THEORY

DAVID BORTHWICK, ALAN MCRAE, AND EDWARD C. TAYLOR

Abstract. Suppose $\Gamma_1$ and $\Gamma_2$ are two convex co-compact co-infinite volume discrete subgroups of $PSL(2, \mathbb{C})$, so that there exists a $C^\infty$ diffeomorphism $\psi : \Omega(\Gamma_1)/\Gamma_1 \to \Omega(\Gamma_2)/\Gamma_2$ that induces an isomorphism $\phi : \Gamma_1 \to \Gamma_2$. For fixed $s \in \mathbb{C}$, let $S_i(s)$ be the scattering operator on $\Omega(\Gamma_i)/\Gamma_i$ ($i = 1, 2$). Define $S_{rel}(s) = S_1(s) - \psi^* S_2(s)$, where $\psi^* S_2(s)$ is the pull-back of $S_2(s)$ to an operator acting on the appropriate complex line bundle over $\Omega(\Gamma_1)/\Gamma_1$. Our main result is: if the operator norm of $S_{rel}(s)$ is $\epsilon$-small, then the $\Gamma_i$ are $K(\epsilon)$-quasi-conformally conjugate and the dilatation $K(\epsilon)$ decreases to 1 as $\epsilon$ decreases to 0.

1. Statement of Results

Geometrically finite Kleinian groups uniformizing infinite volume hyperbolic 3-manifolds exhibit a rich deformation theory due to work of Ahlfors, Bers, Kra, Marden, Maskit, Thurston and others. The purpose of this note is to introduce scattering theory as an analytic tool in the study of the deformations of complete geometrically finite hyperbolic structures. The results in this paper are restricted to a sub-class of geometrically finite Kleinian groups called convex co-compact groups, i.e. those containing no parabolic subgroups.

Assume that $\Gamma$ is a convex co-compact, torsion-free Kleinian group with non-empty regular set $\Omega(\Gamma)$ (see Section 2 for definitions). The compact (possibly disconnected) quotient surface $\Omega(\Gamma)/\Gamma$ is the conformal boundary at infinity of the hyperbolic 3-manifold $M(\Gamma) = \mathbb{H}^3/\Gamma$. To the Laplacian $\Delta$ on $M(\Gamma)$ we can associate a scattering operator acting on sections of certain complex line bundles over the conformal boundary $\Omega(\Gamma)/\Gamma$. These sections are most conveniently described as automorphic forms on $\Omega(\Gamma)$.

For a complex parameter $s$, let $\mathcal{F}_s(\Gamma)$ be the space of automorphic forms of weight $s$ on $\Omega(\Gamma)$ (see Section 3 for the definition). The scattering operator $S(s)$ is a pseudodifferential operator with known singularity mapping $\mathcal{F}_{2-s}(\Gamma) \to \mathcal{F}_s(\Gamma)$. For $\text{Re } s = 1$ we have a natural $L^2$ inner product on $\mathcal{F}_s(\Gamma)$, so we can complete these spaces to form Hilbert spaces.

Date: February 7, 2008.

First author supported in part by NSF grant DMS-9401807. Third author partially supported by a University of Michigan Rackham Fellowship.
Now take two convex co-compact, torsion-free Kleinian groups $\Gamma_{i=1,2}$, with $\Omega(\Gamma_{i=1,2}) \neq \emptyset$. Assume there exists an orientation-preserving diffeomorphism $\psi : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$ that induces an isomorphism $\phi : \Gamma_1 \to \Gamma_2$. Denote by $S_2(s)$ the scattering operator acting on sections over $\Omega(\Gamma_2)/\Gamma_2$. Recall the diffeomorphism $\psi$ descends to a diffeomorphism $\psi : \Omega(\Gamma_1)/\Gamma_1 \to \Omega(\Gamma_2)/\Gamma_2$. Thus we can pull $S_2(s)$ back, via the diffeomorphism $\psi$, to an operator taking acting on sections over $\Omega(\Gamma_1)/\Gamma_1$. Denote this pull-back of $S_2$ by $\psi^*S_2(s)$. Perry [17] shows that if for some $\text{Re } s = 1, s \neq 1$, the operator $S_{rel}(s) = S_1(s) - \psi^*S_2(s)$ is trace-class with respect to the Hilbert space completions of $\mathcal{F}_{2-s}(\Gamma_1)$ and $\mathcal{F}_s(\Gamma_1)$, then $S_{rel}(s) = 0$ and the diffeomorphism is actually a Möbius transformation, i.e. the manifolds $M(\Gamma_i) = \mathbb{H}^3/\Gamma_i$ are isometric.

Our results show that the size of the operator $S_{rel}(s)$ detects how close to being isometric the quotient manifolds are. Recall that the deformation theory of Kleinian groups is based on the notion of quasi-conformal conjugacy (see section 2). Our main result is:

**Main Theorem:** Suppose $\Gamma_{i=1,2}$ are convex co-compact, torsion-free Kleinian groups so that $M(\Gamma_{i=1,2})$ has infinite hyperbolic volume. Let

$$\psi : \Omega(\Gamma_1) \to \Omega(\Gamma_2)$$

be an orientation-preserving $C^\infty$-diffeomorphism conjugating $\Gamma_1$ to $\Gamma_2$. Fix $s \in \mathbb{C} : \text{Re}(s) = 1, s \neq 1$ and let $\epsilon > 0$. There is $K(\epsilon) > 1$ so that $\|S_{rel}(s)\| < \epsilon$ implies that $\Gamma_2$ is a $K(\epsilon)$-quasi-conformal deformation of $\Gamma_1$, where $K(\epsilon) \to 1$ as $\epsilon \to 0$.

The norm $\| \cdot \|$ in the Theorem is the operator norm for the $L^2$ space of sections. Independently, Douady-Earle [8], Reimann [18] and Thurston [19] have shown that each $K$-quasi-conformal deformation of a Kleinian group can be extended to an equivariant $K$-quasi-isometry of $\mathbb{H}^3$, where $K \to 1$ as $K \to 1$. Thus the Main Theorem says that if $S_{rel}(s)$ is small in the operator norm, then $M(\Gamma_1)$ is “nearly isometric” to $M(\Gamma_2)$. In particular, $S_{rel}(s) = 0$ implies that $M(\Gamma_1)$ is isometric to $M(\Gamma_2)$ (a fact which was contained in Perry’s result [17]).

The plan for this paper is as follows: Section 2 and and Section 3 discuss respectively the basics of Kleinian group theory and scattering theory we will be using. Section 4 contains the proof of the Main Theorem, as well as various remarks and conjectures.

**Acknowledgements:** The authors would like to thank Peter Perry for explaining the contents of [17] to the third author while Professor Perry was visiting the SUNY-Stony Brook during the Spring of 1993. We are also indebted to Richard Canary of the University of Michigan for helpful and enjoyable conversations on the contents of this paper.
2. Kleinian Group Basics

We will work interchangeably in both the upper half space model and the ball model of hyperbolic 3-space; both are denoted by $H^3$ and the distance between $x, y \in H^3$ is given by $\rho(x, y)$. Let $Isom_+(H^3)$ denote the group of orientation-preserving isometries of hyperbolic 3-space $H^3$ endowed with the compact-open topology. A Kleinian group $\Gamma$ is a discrete subgroup of $Isom_+(H^3)$. Recall that $Isom_+(H^3)$ has a natural identification with the space of Möbius transformations $\mathbb{M}_\mathbb{C}$

$$\{ g(z) = \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{C}, ad - bc = 1 \}.$$

Thus $\Gamma$ acts on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as a group of conformal homeomorphisms. This action partitions $\hat{\mathbb{C}}$ into two disjoint sets: the limit set and the regular set. Suppose that there is a point $z \in \hat{\mathbb{C}}$ and a neighborhood $U$ of $z$, so that $\gamma(U) \cap U = \emptyset$ for all $\gamma \in \Gamma - \{id\}$. The regular set of $\Gamma$, denoted by $\Omega(\Gamma)$, is the (possibly empty) maximal collection of these points. The limit set $L_\Gamma$ is the complement of the regular set in $\hat{\mathbb{C}}$. The reader is referred to [13] for a discussion of the fundamentals in the theory of Kleinian groups.

Every Kleinian group $\Gamma$ acts discontinuously on $H^3$, and there is a natural geometric model for this action. Choose a point $0 \in H^3$ not fixed by any non-trivial element of $\Gamma$. The Dirichlet polyhedron based at $0$ is the set

$$P_0(\Gamma) = \{ x \in H^3 : \rho(x, 0) \leq \rho(x, \gamma(0)) \ \forall \gamma \in \Gamma \}.$$ 

The intersection of the Euclidean closure of $P_0(\Gamma)$ with $\partial H^3 = \hat{\mathbb{C}}$ is a fundamental domain $D_\Gamma$ for the action of $\Gamma$. We can form the quotient $\Omega(\Gamma)/\Gamma$. If $\Gamma$ is finitely generated, then by the Ahlfors Finiteness Theorem [1] we can conclude that $\Omega(\Gamma)/\Gamma$ consists of a finite collection of Riemann surfaces, each of finite type (i.e. each has finite genus and at most a finite number of punctures).

A Kleinian group is geometrically finite if its action on $H^3$ admits a finite-sided Dirichlet polyhedron. We define $\Gamma$ to be convex co-compact if $P_0(\Gamma)$ is finite-sided, and the Euclidean closure of $P_0(\Gamma) \cap \hat{\mathbb{C}}$ is bounded away (in the chordal metric) from $L_\Gamma$. Specifically, in the case that $\Gamma$ is convex co-compact, torsion-free and $\Omega(\Gamma) \neq \emptyset$, then $M(\Gamma) = (H^3 \cup \Omega(\Gamma))/\Gamma$ is a compact 3-manifold. The interior of the $M(\Gamma)$ has a complete infinite volume hyperbolic structure, and the conformal boundary at infinity $\Omega(\Gamma)/\Gamma$ consists of a finite collection of compact Riemann surfaces. The compactness of $\Omega(\Gamma)/\Gamma$ is a key assumption in this paper.

Geometrically finite Kleinian groups with non-empty regular sets admit a space of deformations of at least 1 complex dimension via the theory of quasi-conformal mappings (see [5]). Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasi-conformal automorphism. The Beltrami coefficient $\mu$ of $f$ is an element in the unit open ball in the complex Banach space $L_\infty(\mathbb{C})$ of equivalence classes of bounded
measurable functions defined by the condition
\[ \frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z} \]  
(2.1) 
a.e. in \( \mathbb{C} \) (here \( \frac{\partial f}{\partial \overline{z}} \) and \( \frac{\partial f}{\partial z} \) are generalized derivatives). The dilatation of \( f \) is defined to be
\[ K(f) = 1 + \frac{||\mu||_{\infty}}{1 - ||\mu||_{\infty}}. \]

Geometrically a \( K \)-qc map has the property that it takes infinitesimal circles to infinitesimal ellipses, such that the ratio of the major to minor axes of the ellipses is bounded above by \( K \). A fundamental result of Ahlfors-Bers [3] says that for any \( \mu \in L_\infty(\mathbb{C}) : ||\mu||_{\infty} < 1 \) there exists a unique quasi-conformal automorphism of the Riemann sphere \( \omega^\mu \) fixing the points \( \{0, 1, \infty\} \), and every quasi-conformal automorphism \( f \) with Beltrami coefficient \( \mu \) is of the form \( f = \alpha \circ \omega^\mu : \alpha \in M^\text{ob}(2) \).

Let \( \Gamma \) be a finitely generated Kleinian group. A quasi-conformal automorphism \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) that satisfies \( f \Gamma f^{-1} \subset M^\text{ob}(2) \) is called a quasi-conformal deformation of \( \Gamma \). Note that if \( w^\mu \) induces a quasi-conformal deformation of \( \Gamma \) and \( \Gamma' = w^\mu \circ \Gamma \circ (w^\mu)^{-1} \), then it is easy to check that \( w^\mu : \Omega(\Gamma) \to \Omega(\Gamma') \), and hence \( w^\mu : L_\Gamma \to L_{\Gamma'} \).

Now let \( \psi \) be a diffeomorphism \( \Omega(\Gamma_1) \to \Omega(\Gamma_2) \) as in Section 1. To extend the mapping \( \psi \) to a mapping of the whole Riemann sphere we will use a foundational result from the theory of Kleinian groups. The following is the version of the Marden Isomorphism Theorem [12] we will use for this purpose. In the statement, \( \mathbf{H}^3 \) denotes the closed 3-ball.

**Theorem 2.1.** Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are Kleinian groups such that \( \Gamma_1 \) is convex co-compact and there exists an orientation-preserving diffeomorphism
\[ \psi : \Omega(\Gamma_1) \to \Omega(\Gamma_2) \]
which induces an isomorphism \( \phi : \Gamma_1 \to \Gamma_2 \). Then \( \psi \) can be extended to a quasiconformal homeomorphism \( \tilde{\psi} \) of the closed ball \( \mathbf{H}^3 \), also inducing \( \phi \).

### 3. Scattering theory

Scattering theory for the Laplacian can be thought of as a functional parametrization of the continuous spectrum according to the asymptotic behavior of eigenfunctions at infinity. The interpretation as scattering appears when one translates eigenfunctions into solutions of either the wave equation or the Schrödinger equation. Given a choice of incoming solution to one of these equations (defined asymptotically), the scattering operator provides the corresponding outgoing solution. Thus it contains all the information that can be detected asymptotically on the propagation of waves through the interior of the manifold.

Scattering theory can be studied on complete Riemannian manifolds with certain regular structure at infinity (see [14] for details and references). In our case, the regularity condition is that in a neighborhood of the conformal boundary at infinity the metric has the standard form \( (dx^2 + dt^2)/t^2 \), \((x, t) \in \mathbb{C} \)
\( \mathbb{R}^2 \times \mathbb{R}_+ \). This is the reason for the requirement that \( \Gamma \) be convex co-compact, which disallows cusps. We hope to extend our results to hyperbolic 3-manifolds with cusps in the future.

The limiting behavior at infinity of eigenfunctions of the Laplacian on \( M(\Gamma) = \mathbb{H}^3 / \Gamma \) is described by sections of complex line bundles on the conformal boundary. In particular, different line bundles over \( \Omega(\Gamma) / \Gamma \) describe incoming and outgoing solutions of the wave equation. We define these line bundles by their spaces of sections, using automorphic forms. An automorphic form of weight \( s \in \mathbb{C} \) on \( \Omega(\Gamma) \) is a function \( f \in C^\infty(\Omega(\Gamma)) \) such that

\[
f(\gamma(x))|\gamma'(x)|^s = f(x), \quad \forall \gamma \in \Gamma.
\]

Here \( |\gamma'(x)| \) is the conformal dilation of the Möbius transformation \( \gamma \) at \( x \in \mathbb{C} \). Denote by \( \mathcal{F}_s(\Gamma) \) the space of all such forms. Each \( \mathcal{F}_s(\Gamma) \) corresponds to the space of smooth sections of a complex line bundle over \( \Omega(\Gamma) / \Gamma \).

The scattering operator \( S(s) \) maps \( \mathcal{F}_{2-s}(\Gamma) \) to \( \mathcal{F}_s(\Gamma) \). Its Schwartz kernel can be written as a series,

\[
S(s; x, y) = \sum_{\gamma \in \Gamma} \frac{|\gamma'(x)|^s}{|\gamma(x) - y|^{2s}},
\]

which converges at least for \( \text{Re } s > 2 \). Note that \( S(s; x, \cdot) \) has weight \( s \), so that

\[
S(s)f(x) = \int_{D_\Gamma} S(s; x, y)f(y)dy,
\]

is well-defined for \( f \) of weight \( 2 - s \), where \( D_\Gamma \) is any choice of fundamental domain for the action of \( \Gamma \) on \( \Omega(\Gamma) \). Since \( S(s; \cdot, y) \) is of weight \( s \) as well, the scattering operator maps weight \( 2 - s \) into weight \( s \) as claimed.

An automorphic form of weight 2 is a density, so we have a natural pairing of \( \mathcal{F}_{2-s}(\Gamma) \) with \( \mathcal{F}_s(\Gamma) \). For \( \text{Re } s = 1 \) complex conjugation takes forms of weight \( s \) to forms of weight \( 2 - s \), so this pairing gives an inner product on \( \mathcal{F}_s(\Gamma) \), \( \text{Re } s = 1 \):

\[
\langle f, g \rangle = \int_{D_\Gamma} \overline{f(x)}g(x)dx.
\]

Accordingly we define the Hilbert space \( \mathcal{H}_\sigma(\Gamma) \) to be the completion in this inner product of \( \mathcal{F}_{1+i\sigma}(\Gamma) \).

Of course, the series given in (3.1) does not necessarily converge for \( \text{Re } s = 1 \), the region of interest. Fortunately, we have the following set of results.

**Theorem 3.1.** (Mazzeo-Melrose [1], Mandowalos [4], Perry [11])

Suppose that \( \Gamma \) is a convex co-compact co-infinite volume discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \). Let \( S(s) \) be the scattering operator mapping \( \mathcal{F}_{2-s}(\Gamma) \) to \( \mathcal{F}_s(\Gamma) \), defined by (3.1) for \( \text{Re } s > 2 \).

1. \( S(s) \) has a meromorphic continuation to \( s \in \mathbb{C} \), with no poles for \( \text{Re } s = 1, s \neq 1 \).
2. \( S(s) \) is an elliptic pseudodifferential operator of order \( 2 - 2 \text{Re } s \).
3. In a disk $D \subset \Omega(\Gamma)$, we can write

$$S(s;x,y) = \frac{1}{|x-y|^{2s}} + \phi(s;x,y), \quad \text{for } x,y \in D$$

where $\phi$ is smooth in $x,y$ and meromorphic in $s$. For $x$ and $y$ lying in distinct neighborhoods, we have simply that $S(s;x,y)$ is smooth in $x$ and $y$.

Remarks

1. These results are obtained by studying the scattering kernel as an asymptotic limit of the resolvent kernel of the Laplacian on $M(\Gamma)$. After showing in this way that $S(s)$ is an elliptic pseudodifferential operator for $\text{Re } s \geq 1$, a parametrix is constructed and used along with a functional relation $S(2-s)S(s) = I$ to obtain the continuation.

2. Note that for $\text{Re } s = 1$, $S(s)$ is a zeroth-order operator. Since the conformal boundary is compact, $S(1+i\sigma)$ extends to a bounded operator on $H_{-\sigma}(\Gamma) \rightarrow H_{\sigma}(\Gamma)$.

3. The third property, which is crucial for our result, says that the principal symbol of the scattering operator for $M(\Gamma)$ is the same as for $H^3$, and in particular independent of $\Gamma$.

4. Proof of the Main Theorem

Recall the setting described in Section 1. $\Gamma_1$ and $\Gamma_2$ are two convex co-compact, torsion-free Kleinian groups with non-empty domains of discontinuity $\Omega(\Gamma_i)$. For convenience, we’ll conjugate $\Gamma_1$ so that $0 \in \Omega(\Gamma_1)$, and choose a fundamental domain $D_{\Gamma_1}$ for the action of $\Gamma_1$ on $\Omega(\Gamma_1)$ so that $0 \in D_{\Gamma_1}$.

Now assume that there exists an orientation-preserving diffeomorphism $\psi : \Omega(\Gamma_1) \rightarrow \Omega(\Gamma_2)$ which induces an isomorphism of $\Gamma_1$ and $\Gamma_2$. It is easy to check that under these conditions the diffeomorphism $\psi$ induces a bijection $\psi^* : F_{s}(\Gamma_2) \rightarrow F_s(\Gamma_1)$, given by

$$\psi^* f(x) = (\det D\psi(x))^{s/2}(f \circ \psi)(x),$$

where $\det D(\psi(x))$ is the Jacobian determinant of the mapping $\psi$ (see Perry [17] section 2).

Let $S_j(s)$, $j=1,2$, denote the scattering operators associated to the two groups. We can use $\psi^*$ to define a pullback the scattering operator $S_2(s)$ to an operator $\psi^* S_2(s) : F_{2-s}(\Gamma_1) \rightarrow F_s(\Gamma_1)$. Thus we can define the relative scattering operator

$$S_{rel}(s) = S_1(s) - \psi^* S_2(s),$$

which we regard by extension as an operator $\mathcal{H}_{-\sigma}(\Gamma_1) \rightarrow \mathcal{H}_\sigma(\Gamma)$, for $s = 1 + i\sigma$.

Our first task is to prove:
Theorem 4.1. For $\text{Re } s = 1, s \neq 1, \varepsilon > 0$ there exists a number $\delta_s(\varepsilon)$ such that

$$\|S_{rel}(s)\| < \varepsilon \implies \psi \text{ is } (1 + \delta_s(\varepsilon))-\text{quasiconformal on } \Omega(\Gamma_1),$$

where $\|\cdot\|$ is the operator norm on $\mathcal{L}(\mathcal{H}_{\sigma}(\Gamma_1), \mathcal{H}_{\sigma}(\Gamma_1))$. We will also observe that as $\varepsilon \downarrow 0$, $\delta_s(\varepsilon) = C\varepsilon + O(\varepsilon^2)$, where $C$ is a constant independent of $s$.

Proof. Fix $s = 1 + i\sigma$. The proof is by contradiction: we assume that $\psi$ is not $(1 + \delta)$-quasiconformal at some point and derive from this a lower bound (a function of $\delta$) on $\|S_{rel}(s)\|$. We then invert this function to obtain $\delta(\varepsilon)$.

Without loss of generality, assume that $\psi$ fails to be $(1 + \delta)$-quasiconformal at $0 \in \Omega(\Gamma_1)$. We can in fact assume this holds true in a neighborhood $|x| < a$ for some $a > 0$. In the course of the proof we will neglect small $a$ error terms, and the final bound will be shown independent of $a$.

The principal symbol of $S_1(s)$ is

$$a_0(x, \xi) = c_\sigma |\xi|^{2\sigma},$$

where $c_\sigma$ is a constant with $|c_\sigma| = 1/(2\sigma)$. The principal symbol of the pullback $\psi^*S_2(s)$ is

$$a'_0(x, \xi) = c_\sigma |A(x)\xi|^{2\sigma}.$$ 

where $A(x)$ is the Jacobian determinant times the inverse of the Jacobian matrix:

$$A(x) = \sqrt{\det D\psi(x) \cdot (D\psi(x))^{-1}}$$

(note $\det A(x) = 1$).

Noting that $\|S_{rel}\|^2 = \|S^*_{rel}S_{rel}\|$, we proceed by analyzing the operator $B = 2\sigma^2S^*_{rel}(s)S_{rel}(s)$ The principal symbol of $B$ is thus

$$b_0(x, \xi) = 1 - \cos \left(\sigma \ln \frac{|A(x)\xi|^2}{|\xi|^2} \right).$$

Let $B_0$ be the pseudodifferential operator with total symbol $b_0(x, \xi)$, so that $B = B_0 + B_1$ where $B_1$ is of order $-1$.

Denote by $\lambda(x)$ and $1/\lambda(x)$ be the two eigenvalues of $A(x)^tA(x)$. We next quote a standard fact relating the quasi-conformal factor to the size of $\lambda$ (see [20], for example).

Lemma 4.2. With $\lambda(x)$ defined as above, the diffeomorphism $\psi$ is $K$-quasi-conformal if and only if $K = \sup \lambda(x)$.

As outlined above, we assume that $\psi$ is not $(1 + \delta)$-quasiconformal in a neighborhood $\{|x| < a\} \subset \Omega(\Gamma_1)$. We also assume that $\{|x| < a\} \subset \mathcal{D}_{\Gamma_1}$. Our hypothesis for the proof by contradiction thus amounts to the assumption that $\lambda(x) > 1 + \delta$ for all $|x| < a$.

The lower bound on $\|S_{rel}\|$ is obtained from this assumption by probing with particular $L^2$ states. The first serves to localize near $x = 0$, and is defined in $\mathcal{D}_{\Gamma_1}$ by

$$\phi_1(x) = \frac{C_1}{a} \Theta(a - |x|),$$

where $C_1$ is a constant.
where $\Theta$ is the step function and $C_1$ is a constant chosen so that $\|\phi_1\| = 1$.

For the second, we would like to choose
\[
\phi_2(x) = \frac{C_2}{a} e^{-|x|^2/2a^2},
\]
where $\xi_0$ is fixed, so that
\[
\hat{\phi}_2(\xi) = C_2' e^{-a^2|\xi|^2/2}.
\]
But as this does not give a smooth section, we must multiply the $\phi_2$ given in (4.1) by a smooth cutoff of $\phi_2$ near the boundary of $D_{\Gamma_1}$. This produces an error term in the $\hat{\phi}_2$ given in (4.2), which is uniformly $O(a^\infty)$ and decreases rapidly in $\xi$. Thus we may safely neglect this error by choosing $a$ sufficiently small.

Observe that $|\langle S_{rel}(s)\phi_1, S_{rel}(s)\phi_2 \rangle| \leq \|S_{rel}(s)\|^2$. Recalling that $B_0 + B_1 = 2\sigma^2 S_{rel}(s)^* S_{rel}(s)$, this means
\[
\|S_{rel}(s)\|^2 \geq \frac{1}{2\sigma^2} \langle \phi_1, (B_0 + B_1)\phi_2 \rangle
\]
(4.3)

Using the fact that $B_1$ is of order $-1$ it is straightforward to produce a bound
\[
\|B_1\phi_2\| \leq Ca,
\]
reflecting the fact that $\hat{\phi}_2$ spreads out as $a$ becomes small. Since $a$ can be made arbitrarily small, this combines with (4.3) to give us
\[
\|S_{rel}(s)\|^2 \geq \frac{1}{2\sigma^2} |\langle \phi_1, B_0\phi_2 \rangle|
\]

For ease of exposition in the following integral formulas, let $D = D_{\Gamma_1}$. We proceed from the explicit formula
\[
\langle \phi_1, B\phi_2 \rangle = \int\int_D \phi_1(x)b_0(x,\xi)e^{ix\cdot\xi}\hat{\phi}_2(\xi)\,dx\,d\xi
\]
Writing $\xi = r\eta$ where $|\eta| = 1$
\[
\langle \phi_1, B\phi_2 \rangle = C \int\int_D \Theta(a - |x|)b_0(x,\eta)r e^{ir(x\cdot\eta)}e^{-a^2r^2/2}\,dx\,dr\,d\eta.
\]
Noting that $|\langle \phi_1, B\phi_2 \rangle| \geq |\text{Re}\langle \phi_1, B\phi_2 \rangle|$, consider
\[
\text{Re}\langle \phi_1, B\phi_2 \rangle = C \int\int_D \Theta(a - |x|)b_0(x,\eta)r \cos(rx\cdot\eta)e^{-a^2r^2/2}\,dx\,dr\,d\eta.
\]
We extract that $r$ integral:
\[
\int_0^\infty r \cos(rx\cdot\eta)e^{-a^2r^2/2}\,dr = C \frac{1}{a^2} \int_0^\infty r \cos\left(r \frac{x\cdot\eta}{a}\right)e^{-r^2/2}\,dr.
\]
Since $(x\cdot\eta) \leq a$, we have
\[
\int_0^\infty r \cos\left(r \frac{x\cdot\eta}{a}\right)e^{-r^2/2}\,dr \geq \frac{1}{4},
\]
so that
\[
|\langle \phi_1, B\phi_2 \rangle| \geq C \frac{1}{a^2} \int\int_D \Theta(a - |x|)b_0(x,\xi)\,dx\,d\eta.
\]
Now consider the \( \eta \) integration. By an appropriate change of variables
we can write \( |A(x)\eta|^2 = \lambda(x) \cos^2 \theta + \lambda(x)^{-1} \sin^2 \theta \) and \( d\eta = d\theta \). One can
check that as a function of \( \lambda > 1 \), the integral
\[
|A(x)\eta|_2^2 = \lambda(x) \cos^2 \theta + \lambda(x)^{-1} \sin^2 \theta
\]
is monotonically increasing. We thus have
\[
|\langle \phi_1, B\phi_2 \rangle| \geq C f_\sigma(1 + \delta)
\]
Note that this bound is uniform in \( a \), justifying our earlier assumptions.

To summarize, we have shown that if \( \psi \) is not \((1 + \delta)\)-quasiconformal
then \( \|S_{\sigma \eta}(s)\|^2 > C\sigma^{-2}f_\sigma(1 + \delta) \). It is easy to see that for small \( \delta \) (with \( \sigma \)
fixed), \( f_\sigma(1 + \delta) = C\sigma^2\delta^2 + O(\delta^3) \). This completes the proof of Theorem 4.1.

\[\square\]

**Proof of the Main Theorem.** We have shown that \( \psi \) is a \( K(\varepsilon)\)-quasi-
conformal mapping defined on \( \Omega(\Gamma_1) \), where \( K(\varepsilon) = 1 + \delta(\varepsilon) \). By assumption
\( \psi \) induces an isomorphism \( \phi : \Gamma_1 \to \Gamma_2 \). So by the Marden Isomorphism
Theorem (Theorem 2.1), we can extend \( \psi \) to a quasi-conformal homeomorphism of the closed ball \( \hat{H}^3 \) that induces \( \phi \).

Recall that the limit set \( L_{\Gamma_1} \) has measure zero \([1]\), so we can assume
the Beltrami coefficient of the extended mapping to be zero on \( L_{\Gamma_1} \). Thus
the extension of \( \psi \) is a \( K(\varepsilon)\)-quasi-conformal deformation taking \( \Gamma_1 \) to \( \Gamma_2 \).

\[\square\]

We can use the Main Theorem to make the following observation concerning the Hausdorff dimension of the limit sets \( L_{\Gamma_1} \) and \( L_{\Gamma_2} \). For a set \( E \subset \hat{C} \) let 
\( D(E) \) be the *Hausdorff dimension* of \( E \). Work of Gehring-Väisälä \([9]\)
and Astala \([4]\) demonstrates that for a set \( E \subset \hat{C} \), and a \( K \)-quasi-conformal
mapping \( f \) with \( E \) in its domain, then the set \( f(E) \) has Hausdorff dimension
bounded above and below by
\[
\frac{2D(E)}{2K + (K-1)D(E)} \leq D(f(E)) \leq \frac{2KD(E)}{2 + (K-1)D(E)}.
\]
We recall that the quasi-conformal conjugacy \( \psi : \hat{C} \to \hat{C} \) taking \( \Gamma_1 \) to \( \Gamma_2 \)
has the property that \( L_{\Gamma_2} = \psi(L_{\Gamma_1}) \). Thus we have the following corollary:

**Corollary 4.3.** Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are convex co-compact groups satisfying
the conditions given in the Main Theorem. Then there exists a \( \nu(\varepsilon) > 0 \) so that
\[
|D(L_{\Gamma_1}) - D(L_{\Gamma_2})| < \nu(\varepsilon),
\]
where \( \nu(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).
Remarks:

1. It is natural to ask whether a small quasi-conformal deformation implies that the relative scattering operator is small. In the discussion below we assume familiarity with the definition of the Poincaré series of a Kleinian group $\Gamma$ and with the exponent of convergence $\delta(\Gamma)$ of this series; we refer the reader to [15] for discussion of these matters.

In [6], using entirely two-dimensional techniques, we have been able to show:

**Theorem:** Fix $s \in \mathbb{C} : s = 1 + \sigma i, \sigma \neq 0$, and suppose that $\Gamma_1$ is a torsion-free non-elementary convex co-compact Kleinian group so that $\delta(\Gamma_1) < 1$. Then for any $\epsilon > 0$ and for all dilatations $K$ sufficiently close to 1, each $K$-quasi-conformal deformation of $\Gamma_1$ has the property that

$$||S_{rel}(s)|| < \epsilon,$$

where $|| \cdot ||$ is the operator norm on $\mathcal{L}(\mathcal{H}_{-\sigma}(\Gamma_1), \mathcal{H}_{\sigma}(\Gamma_1))$.

The condition $\delta(\Gamma_1) < 1$ is topologically restrictive: under the assumptions given to $\Gamma_1$ in the theorem above, results in [6] imply that $M(\Gamma_1)$ is the interior of a solid handlebody. However, we believe that a similar result can be shown for two convex co-compact groups $\Gamma_i$ with no assumptions on the exponents of convergence of the Poincaré series. This would involve a 3-dimensional approach using the asymptotic geometry of $M(\Gamma)$ and the limiting properties of the integral kernel of the resolvent of the Laplace operator on $M(\Gamma)$ (see [16]).

2. Corollary 4.3 shows that for convex co-compact Kleinian groups the size of the relative scattering operator contains information concerning the distortion of the Hausdorff dimension of the limit sets. We are thus motivated to ask: *Can the Hausdorff dimension of the limit set be recovered from scattering data?*

**References**


