DETERMINANTS OF LAPLACIANS AND ISOPOLAR METRICS
ON SURFACES OF INFINITE AREA

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Abstract. We construct a determinant of the Laplacian for infinite-area surfaces which are hyperbolic near infinity and without cusps. In the case of a convex co-compact hyperbolic metric, the determinant can be related to the Selberg zeta function and thus shown to be an entire function of order two with zeros at the eigenvalues and resonances of the Laplacian. In the hyperbolic near infinity case the determinant is analyzed through the zeta-regularized relative determinant for a conformal metric perturbation. We establish that this relative determinant is a ratio of entire functions of order two with divisor corresponding to eigenvalues and resonances of the perturbed and unperturbed metrics. These results are applied to the problem of compactness in the smooth topology for the class of metrics with a given set of eigenvalues and resonances.

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1. Introduction

Determinants of Laplacians provide an important, non-local spectral invariant which plays a key role in the analysis of isospectral sets of compact surfaces. The goal of this paper is to develop the determinant of the Laplacian for complete non-compact hyperbolic surfaces and their generalizations, as a tool in the spectral theory of these spaces. Such surfaces have infinite metric volume and at most finitely many $L^2$-eigenvalues, so that ‘most’ of the geometric information is contained in the resonances (poles of the meromorphically continued resolvent). Thus it is natural to consider isopolar classes of manifolds, those with the same eigenvalues and resonances, and to investigate the restrictions imposed on their geometry.

For compact and finite-volume surfaces determinants are well-understood objects (see, for example [12, 13, 31, 32, 35, 37]). In the infinite-volume case there are several important points of contrast. First and foremost, it is not feasible to define the determinant through zeta-function regularization. Instead, we adopt a ‘Green’s function method,’ i.e. we define the determinant through the formal identity

$$\left( \frac{d}{dz} \right)^2 \log \det(A + z) = -\text{tr}(A + z)^{-2},$$

valid for finite matrices $A$. By regularizing the trace of the square of the resolvent, one can produce a function $D(s)$, formally equal to $\det(\Delta + s(s - 1))$.

The second issue is that the determinant is not a priori a ‘spectral’ invariant; Indeed, a substantial part of our work will be to show that the eigenvalues and resonances do determine $D(s)$ up to finitely many parameters.

A final major distinction to the compact case is that $\det(\Delta)$ so defined is a proper function only on certain subsets of the moduli space of the underlying topological surface (compare [21] where a similar phenomenon is analyzed for compact surfaces with boundary). This lack of properness over the entire moduli limits the determinant’s usefulness in determining whether or not isopolar classes are compact.

Throughout this paper we will consider only hyperbolic surfaces that are convex co-compact, which means complete, finite topological type, infinite area, and without cusps. For such hyperbolic metrics, the results of Patterson-Perry [33] allow us to evaluate $D(s)$ in terms of Selberg’s zeta function $Z_{\tau}(s)$ (our Theorem 3.2), which is an entire function of order two. We thereby deduce that $D(s)$ has real zeros in the half-plane $\text{Re}(s) > 1/2$ corresponding to eigenvalues and zeroes corresponding to resonances in the half-plane $\text{Re}(s) < 1/2$ and possibly the special value $\text{Re}(s) = 1/2$.

Let $\bar{X}$ be a compact manifold with boundary, and $\rho$ be a defining coordinate for the boundary $\partial \bar{X}$, i.e. $\rho \geq 0$, $\partial \bar{X} = \{ \rho = 0 \}$, and $d\rho|_{\partial \bar{X}} \neq 0$. A conformally compact metric is a metric on $X$ the form $g = \rho^{-2} \bar{g}$, where $\bar{g}$ is a smooth metric on $\bar{X}$. The metric is called asymptotically hyperbolic if $|dx|_{\bar{g}}|_{\partial \bar{X}} = 1$, which implies that Gaussian curvature of $g$ approaches $-1$ at $\partial X$. (Note that this definition precludes cusps; the hyperbolic metrics which are conformally compact are precisely the convex co-compact ones.) Asymptotically hyperbolic is the level of generality of the Mazzeo-Melrose [28] parametrix construction and proof of meromorphic continuation of the resolvent, although some extension to the conformally compact case is possible [4]. For an asymptotically hyperbolic metric one can still define the determinant through the resolvent as above, but there is no analogous zeta function theory for this case. Indeed, it is possible that the resolvent will have infinite-rank

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poles at \(-\frac{1}{2}N_0\), in which case \(D(s)\) would fail to be entire. To avoid this, as well as other complications, we specialize to metrics that are hyperbolic near infinity (always assumed without cusps), meaning of constant curvature \(-1\) outside some compact set (same context as [16]).

For metrics hyperbolic near infinity, in addition to the determinant defined as above we can also construct a relative determinant based on conformal metric perturbation. Recent results of Mazzeo-Taylor [27] imply that such a metric can be 'uniformized' as \(g = e^{2\varphi} \tau\), where \(\tau\) is a convex co-compact hyperbolic metric on \(X\) and \(\varphi \in \rho^2 C^\infty(\bar{X})\). The decay of \(\varphi\) at the boundary allows the definition of a relative determinant of the Laplacians \(\Delta_g\) and \(\Delta_\tau\), by zeta function regularization as in Müller [30].

The heat kernel definition of the relative zeta function is
\[
\zeta(w, s) = \frac{1}{\Gamma(w)} \int_0^\infty t^w e^{-ts} \text{tr}[e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau}] \frac{dt}{t},
\]
where \(\hat{\Delta}_g\) is the pull-back of \(\Delta_g\) to \(L^2(X, d\tau)\), and our convention is that the Laplacian be a positive operator. This converges for \(\text{Re}(w) > 1\) and \(\text{Re}(s(s-1)) > -\mu\), where \(\mu > 0\) is the joint infimum of the spectra of the two Laplacians. By analytic continuation to \(w = 0\) we may define
\[
D_{g,\tau}(s) = -\frac{d}{dw} \zeta_w(w, s) \bigg|_{w=0},
\]
for \(\text{Re}(s(s-1)) > -\mu\).

One would expect the relative determinant to have zeroes at the eigenvalues and scattering poles of \(g\), and poles at those of \(\tau\). In fact we can fully characterize the relative determinant as a meromorphic function. Let \(P_g(s)\) and \(P_\tau(s)\) be Hadamard products formed from the eigenvalues and resonances, respectively, of \(g\) and \(\tau\) (see (5.1) for a precise definition).

**Theorem 1.1.** Suppose \(g\) is a metric hyperbolic near infinity, uniformized as \(g = e^{2\varphi} \tau\). The relative determinant \(D_{g,\tau}(s)\) extends to a meromorphic function of the form
\[
D_{g,\tau}(s) = e^{q(s)} P_g(s)/P_\tau(s).
\]
The polynomial \(q(s)\) has degree at most two and is determined by the eigenvalues and resonances of \(\Delta_g\).

The proof of Theorem 1.1 consists of three steps. First, we compute the divisor of \(D_{g,\tau}(s)\) as follows. We determine its zeros in \(\text{Re}(s) \geq 1/2\) directly and obtain a functional equation of the form
\[
e^{h(s)} \frac{D_{g,\tau}(s)}{D_{g,\tau}(1-s)} = \text{det}(S_{g,\tau}(s)),
\]
where \(S_{g,\tau}(s)\) is the relative scattering operator, and \(h(s)\) is a polynomial of degree at most two. We then use results of Guillopé-Zworski [17] on determinants of scattering operators to compute the divisor of the meromorphic function \(\text{det}(S_{g,\tau}(s))\). Secondly, we show that \(D_{g,\tau}(s)\) is a quotient of entire functions of order at most four by using estimates on the relative zeta function together with constructive estimates on the resolvent \(R_g(s)\) proved in [17] (this is another point where the restriction to hyperbolic near infinity is required). Thirdly, we check that \(D_{g,\tau}(s)\) is entire of order two and prove the statement about \(q(s)\) by studying the asymptotics of \(\log D_{g,\tau}(s)\) as \(\text{Re}(s) \to \infty\).
In the compact case there is a ‘Polyakov’ formula expressing the relative determinant for a conformal perturbation in terms of the conformal parameter, due to Polyakov [34] and Alvarez [1]. Since the proof is based on the zeta regularization, the extension to our situation is quite straightforward.

**Proposition 1.2.** Suppose \( g \) is an asymptotically hyperbolic metric with \( K(g) + 1 = O(\rho^2) \), uniformized as \( g = e^{2\varphi} \tau \). Then

\[
\log D_{g,\tau}(1) = -\frac{1}{6\pi} \int_X \left( \frac{1}{2} |\nabla_{\tau} \varphi|^2 - \varphi \right) d\tau.
\]

With these tools in place, we turn to the isopolar problem. Consider a topological surface \( X \) of signature \((h,M)\), i.e. \( X \) is a surface with genus \( h \) having \( M \) discs removed. The diffeomorphism classes are determined by the signature, and the Euler characteristic is given by

\[
\chi(X) = 2 - 2h - M.
\]

Let \( \tau \) be a hyperbolic metric on \( X \) which makes \( X \) a complete Riemannian manifold whose ideal boundary consists of \( M \) circles. The surface \( (X,\tau) \) takes the form

\[
\hat{X} \sqcup F_1 \sqcup \cdots \sqcup F_M.
\]

Here \( \hat{X} \), the convex core of \( (X,\tau) \), is a convex compact manifold of genus \( h \) with geodesic boundary consisting of \( M \) closed geodesics which we denote by \( \gamma_1, \ldots, \gamma_M \). Letting \( \ell_i \) be the geodesic length of \( \gamma_i \), the \( F_i \) are hyperbolic funnels isometric to the half-cylinder \((0,\infty) \times S^1_\rho \) with metric

\[
ds^2 = dr^2 + \ell_i^2 \cosh^2 r \, d\theta^2.
\]

The \( F_i \) are glued to \( \hat{X} \) along the bounding geodesics.

By studying the asymptotics of \( D_{g,\tau}(s) \) as \( \text{Re}(s) \to \infty \), we shall prove the following:

**Proposition 1.3.** Let \( g \) be a metric hyperbolic near infinity on \( X \). The scattering poles and eigenvalues of \( g \) determine the Euler characteristic of \( X \). Thus the set of all surfaces \((X,g)\) with given eigenvalues and scattering poles contains at most finitely many diffeomorphism types.

In fact, the eigenvalues and scattering poles of \( g \) also determine the relative heat invariants for the pair \((g,\tau)\), which will be defined in \([4]\).

In view of Proposition 1.3 we will fix a diffeomorphism type \((h,M)\) and a model manifold \( X \) with compactification \( \hat{X} \). Suppose that \( X \) carries a sequence of isopolar metrics \( g_n \). We wish to establish compactness of the isopolar set by showing that a subsequence of the \( g_n \)'s converges (modulo diffeomorphism) in a \( C^\infty \) topology. Roughly the strategy is as follows: uniformize \( g_n = e^{2\varphi_n} \tau_n \), and define relative determinants \( D_{g_n,\tau_n}(s) \). Note that \( D_{g_n,\tau_n}(s) \) is not determined by the eigenvalues and resonances of \( g_n \), and therefore not independent of \( n \) (in contrast to the Osgood-Phillips-Sarnak case [32]). However, the relative heat invariants do turn out to be ‘spectral’ invariants, and they give uniform estimates on \( \varphi_n \) and its derivatives, expressed as integrals over \( \tau_n \). These estimates may be combined with Theorem 1.1 and Proposition 1.2 to give uniform control over \( Z_{\tau_n}(1) \).

We will demonstrate in Appendix A that the evaluation of the Selberg zeta function at 1, \( \tau \mapsto Z_{\tau}(1) \), is not a proper function on the moduli space of \( X \). Let \( \hat{X}_\tau \) denote the convex core of \((X,\tau)\) and let \( \ell(\partial \hat{X}_\tau) \) denote the sum of the lengths
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of the bounding geodesics. If we restrict to a subset of moduli space where \( \ell(\partial \hat{X}_\tau) \) is bounded above, then \( Z_\tau(1) \) is a proper function. With this restriction, then, the uniform control of \( Z_\tau_n(1) \) does imply convergence of a subsequence of the \( \tau_n \).

From this point on the strategy is identical to Osgood-Phillips-Sarnak, and the final result is:

**Theorem 1.4.** Let \( g_n \) be a sequence of isopolar metrics of the form \( g_n = e^{2\varphi_n} \tau_n \), where each \( \varphi_n \) is supported in \( \hat{X}_{\tau_n} \) and such that \( \ell(\partial \hat{X}_{\tau_n}) \) is uniformly bounded in \( n \). Then there is a subsequence of the \( g_n \) which converges, modulo diffeomorphisms of \( X \), in the topology of \( \rho^{-2}C^\infty(\bar{X}, S^2) \), to a non-degenerate limiting metric in the same isopolar class.

Here \( S^2 \) denotes the bundle of symmetric 2-tensors. We note that convergence in the weaker topology of \( C^\infty(X; S^2) \) would not guarantee that the limit metric remains in the isopolar class.

**Remarks.**

1. Examples of isopolar infinite-volume surfaces were given in Remark 2.3 of [17], based on the transplantation method of Bécard [3]. Appendix D (contributed by Robert Brooks) presents some explicit cases that may be constructed by Sunada methods.

2. Our current methods require a priori information concerning hyperbolic uniformization. One might hope that \( \ell(\partial \hat{X}_{\tau_n}) \) could be controlled uniformly by bounding the perimeter of the convex core of \( g_n \), but we are not aware of any such comparison a priori. Our methods do not yield a comparison of this type because all of the polar invariants are expressed in terms of the measures \( d\tau_n \).

Until convergence of a subsequence of the \( \tau_n \) is established, the relative heat invariants and the Polyakov formula give no uniform information about the \( \varphi_n \)'s. The uniform bound on \( \ell(\partial \hat{X}_{\tau_n}) \) must therefore be imposed explicitly to obtain the compactness of the \( \tau_n \) sequence.

3. Assuming a convergent sequence of \( \tau_n \)'s, the invariants coming from the relative determinant give uniform \( H^m(X) \) bounds on \( \varphi_n \) for all \( m \). This does not imply compactness of the sequence \( \{\varphi_n\} \) in a \( C^\infty(X) \) topology, which is the reason for the restriction on the support of \( \varphi_n \).

As a special case, one can consider isopolar classes of convex co-compact hyperbolic metrics. In Theorem 3.1 we will prove an analog of Huber’s Theorem on the equivalence of the length spectrum and the set of eigenvalues and resonances. Then in Theorem A.5 we will show that the set of convex co-compact manifolds with the same length spectrum is finite. Together these imply:

**Theorem 1.5.** Let \( R > 0 \). Each set of isopolar convex co-compact hyperbolic surfaces with \( \ell(\partial X_\tau) < R \) is finite.

This theorem brings up an interesting question. For a purely hyperbolic metric \( \tau \), can an upper bound for \( \ell(\partial X_\tau) \) be deduced from knowledge of eigenvalues and resonances? This is simpler than what would be required to generalize the Theorem 1.4. For a 1-holed torus (\( h = 1, M = 1 \)), Buser-Semmler [10] show that there are no non-isometric surfaces with the same length spectrum, so the answer is affirmative in the particular case. No other cases appear to be known.

The results here complement the recent paper of Hassell-Zelditch [18] where determinants of Laplacians on exterior planar domains in Euclidean space are defined.
and a compactness result for exterior domains with the same scattering phase is proved. Roughly and informally, the scattering phase is determined up to finitely many parameters by the scattering poles, although this statement is difficult to make precise owing to the lack of a sharp Poisson formula for resonances in this setting (see [39] for the best known results).

The plan of this paper is as follows. In §2 we briefly review the spectral and scattering theory for the Laplacian on asymptotically hyperbolic surfaces, and then define the determinant of the Laplacian. This determinant is analyzed in the hyperbolic case in §3. In §4 we define and analyze the relative determinant for a conformal metric perturbation, and prove Proposition 1.2. Theorem 1.1 is proved in §5. Finally, in §6 we prove the compactness theorems for isopolar metrics. Appendix A contains the discussion of the properness of $Z_\tau(1)$ as a function on moduli space, based on Theorem A.4, a generalization of Bers’ theorem for pants decompositions of hyperbolic surfaces with geodesic boundary. Appendices B and C contain certain technical facts needed elsewhere in the paper. Finally, Appendix D, contributed by Robert Brooks, discusses examples of isopolar surfaces arising from the Sunada construction.

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2. Definition of the determinant

We begin by recalling some basic facts about the spectral theory of asymptotically hyperbolic surfaces. For such a metric $g$, the Laplacian $\Delta_g$ has at most finitely many eigenvalues in $[0, \frac{1}{4})$ (see [22] for constant curvature and [24, 25] for variable curvature) and absolutely continuous spectrum of infinite multiplicity in $[\frac{1}{4}, \infty)$ with no embedded eigenvalues (see e.g. [22] for constant curvature and [26] for the variable curvature). Thus the resolvent $(\Delta_g - z)^{-1}$ is a meromorphic operator-valued function in the cut plane $\mathbb{C} \setminus [\frac{1}{4}, \infty)$. Introducing the natural hyperbolic spectral parameter $s$, we write

$$R_g(s) = (\Delta_g + s(s - 1))^{-1}.$$ 

Considered as a map from $L^2(X)$ to itself, $R_g(s)$ is then meromorphic in the half-plane $\text{Re}(s) > \frac{1}{2}$ with poles at real numbers $\zeta > \frac{1}{2}$ for which $\zeta(1 - \zeta)$ is an eigenvalue of the Laplacian. Mazzeo and Melrose [22] showed that, when viewed as a map from $C^\infty_0(X)$ to $C^\infty(X)$, $R_g(s)$ admits a meromorphic extension to the complex plane. Singularities of $R_g(s)$ with $\text{Re}(s) \leq \frac{1}{2}$ are called resonances (or scattering resonances). We denote the full set of poles of $R_g(s)$ (both eigenvalues and resonances) by $\mathcal{R}_g$. A multiplicity can be assigned to each point $\zeta \in \mathcal{R}_g$ as follows. About each such $\zeta$, the resolvent $R_g(s)$ has a Laurent expansion with finite polar part of the form

$$\sum A_j(s - \zeta)^{-j}$$

where the $A_j$ are finite-rank operators; we define the multiplicity, $m_\zeta$, of a point $\zeta \in \mathcal{R}_g$ to be

$$m_\zeta = \dim(\oplus_j \text{Ran}(A_j)).$$
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(For an $L^2$ eigenvalue this definition coincides with the usual notion of multiplicity.) For convenience, we will assume that poles are listed in $\mathcal{R}_g$ according to their multiplicity. In case $g$ has constant curvature $-1$, the existence of infinitely many scattering resonances follows from Example 6, p. 856 and Remark 3, p. 851 of [36].

A ‘determinant of the Laplacian’ should be an entire function $D(s) = \det(\Delta_g + s(s - 1))$ with zeros of multiplicity $m_\zeta$ at each point $\zeta \in \mathcal{R}_g$. To motivate the definition we choose, consider the determinant $D_A(s) = \det(A + s(s - 1))$ where $A$ is a finite dimensional matrix. One then has

$$\left(\frac{1}{2s - 1} \frac{d}{ds}\right)^2 \log D_A(s) = - \text{tr} [(A + s(s - 1))^{-2}]$$

which suggests that $D(s)$ may be defined by replacing the right-hand side by a suitable ‘trace’ of $R_g(s)^2$.

Although $R_g(s)^2$ is not trace-class, its kernel is continuous. Moreover, $R_g(s)$ belongs to an algebra of pseudodifferential operators, the 0-pseudodifferential operators on $X$ (see [25] or [29]), for which a natural renormalized trace is defined. To describe it, recall from the introduction that $X$ is assumed to be conformally compact with respect to a boundary defining function $\rho$. If $P$ is a 0-pseudodifferential operator on $X$ with continuous kernel $K_P$ (with respect to Riemannian measure), then

$$0\text{-tr}(P) = \text{FP}_{\varepsilon \downarrow 0} \left( \int_{\rho \geq \varepsilon} K_P(x, x) \, dg(x) \right)$$

(2.1)

where $\text{FP}_{\varepsilon \downarrow 0}(\cdot)$ denotes the Hadamard finite part. (The structure of the 0-calculus guarantees that the argument has an asymptotic expansion in $\varepsilon$ as $\varepsilon \to 0$, and the Hadamard finite part is simply the constant term in this expansion.)

In a similar way one can define the 0-integral of any smooth function on $\bar{X}$, and the 0-volume of $X$ is just the 0-integral of 1. Note that all of these definitions are dependent on the choice of $\rho$. The 0-integral of a smooth function depends on the 1-jet of $\rho$ restricted to $\partial X$.

We will define a determinant $D_g(s)$ (up to two free parameters plus dependence on the defining function) by the equation

$$\left(\frac{1}{2s - 1} \frac{d}{ds}\right)^2 \log D_g(s) = - 0\text{-tr} [R_g(s)^2].$$

(2.2)

If $g$ is hyperbolic near infinity, then the poles of $R_g(s)$ are known to have finite-rank and one can hope that the function $D_g(s)$ would be entire. We will see that it is entire provided that an appropriate defining function is used to define the 0-trace. If $g$ is only asymptotically hyperbolic then infinite-rank poles at $-\frac{1}{2}N_0$ cannot be ruled out.

3. PROPERTIES OF THE DETERMINANT IN THE HYPERBOLIC CASE

In this section we will develop the theory of the determinant for the case of a convex co-compact hyperbolic metric $\tau$ on $X$. Recall that the Selberg zeta function
$Z_\tau(s)$ is defined for $\Re(s) > 1$ as a product over primitive closed geodesics $\gamma$ of $X$:

$$Z_\tau(s) = \prod_{\gamma} \prod_{k=0}^{\infty} \left[ 1 - e^{-(s+k)\ell(\gamma)} \right]$$

(3.1)

where $\ell(\gamma)$ is the length of $\gamma$. It is known from [33] (see Theorems 1.5 and 1.6) that $Z_\tau(s)$ is an entire function of order two with zeros at the eigenvalues and scattering resonances together the topological zeros of multiplicity $-2(s+1)\chi(X)$ at $s \in -\mathbb{N}_0$, where $\chi(X)$ is the Euler characteristic of $X$. Let $Z_\infty(s)$ be the function (cf. Sarnak [33])

$$Z_\infty(s) = \left[ \frac{(2\pi)^s \Gamma_2(s)^2}{\Gamma(s)} \right]^{-\chi(X)}.$$

(3.2)

Here $\Gamma_2(s)$ is Barnes’ double gamma function [4], defined by the Hadamard product

$$\frac{1}{\Gamma_2(s+1)} = (2\pi)^{s/2} e^{-s/2} \prod_{k=1}^{\infty} \left( 1 + \frac{s}{k} \right)^k e^{-s^2/k}$$

with $\gamma$ Euler’s constant. The function $Z_\infty(s)$ cancels the topological zeros of $Z_\tau(s)$, so that their product is an entire function of order two. It is easily seen that

$$\frac{1}{\chi(X)(2s-1)} Z_\infty(s) = -1 - \frac{1}{s} - \gamma + \sum_{k=1}^{\infty} \frac{s}{k(s+k)}.$$

(3.3)

The Hadamard product for the eigenvalues and resonances is

$$P_\tau(s) = \prod_{\zeta \in \mathcal{K}_\tau} \left( 1 - \frac{s}{\zeta} \right)^{m_\zeta} e^{m_\zeta\left( -\frac{s}{\zeta} + \frac{\pi^2}{6\zeta^2} \right)}.$$

(3.4)

By the characterization of $Z_\tau(s)$ in [33], we have

$$Z_\tau(s)Z_\infty(s) = e^{q(s)}P_\tau(s),$$

(3.5)

for some polynomial $q(s)$ of degree two. In particular, the length spectrum determines the eigenvalues and resonances. Asymptotic analysis of (3.5) leads to the following infinite-volume analog of Huber’s Theorem:

**Theorem 3.1.** The eigenvalues and resonances of $\Delta_\tau$ (with multiplicities) determine both $\chi(X)$ and the function $Z_\tau(s)$. Thus for a convex co-compact hyperbolic metric the eigenvalues and resonances determine the length spectrum and vice versa.

**Proof.** The expansion of $\log Z_\infty(s)$ was used prominently by Sarnak [33] and is based on classical results of Barnes [4] for the double gamma function. For $\Re(s) \to \infty$ we have

$$\log Z_\infty(s) \sim \chi(X)(\frac{1}{2} \log 2\pi + \frac{1}{4} - 2\zeta'(-1) - \left( \frac{1}{2}s(s-1) - \frac{1}{6} \right) \log s(s-1)$$

$$+ \frac{3}{2}s(s-1)) + \sum_{l=1}^{\infty} c_l|s(s-1)|^{-l}.$$

(3.6)

In particular, this expansion has a term of the form

$$-\chi(X)(\frac{1}{2}s(s-1) - \frac{1}{6}) \log s(s-1).$$

(3.7)

It is clear from (3.1) that $\log Z_\tau(s) = O(e^{-\ell_0})$ as $\Re(s) \to \infty$, where $\ell_0$ is the length of the shortest closed geodesic on $X$. Thus the left-hand side of (3.7) has an asymptotic expansion with a term of the form (3.7). This could not possibly be
canceled by \( q(s) \) on the right-hand side, so \( \chi(X) \) is determined by \( P_\tau(s) \) and hence by the set of eigenvalues and resonances.

Once \( \chi(X) \) is known, \( q(s) \) is the only unknown in the asymptotic expansion of \( \log P_\tau(s) \), so it too must be determined by the eigenvalues and resonances. Note from above that \( Z_\infty(s) \) depends only on \( \chi(X) \). Thus by (3.5), \( Z_\tau(s) \) is fixed by the eigenvalues and resonances.

The proof that the length spectrum may be extracted from \( Z_\tau(s) \) is completely analogous to the compact case. For example, one may define \( \ell_0 \) as the unique number \( \omega \) such that for real \( s < -\infty \)
\[
\lim_{s \to -\infty} e^{\omega s} \log Z_\tau(s) < 0.
\]
Then terms with \( \ell_0 \) are removed from the product, the same approach determines \( \ell_1 \), etc.

In order to connect the determinant to the zeta function we must be careful about the definition of the 0-trace. On a hyperbolic surface one may specify a natural class of defining function by requiring that the 0-volume of the funnels \( F_j \) equal zero. For instance, in the model metric (1.1) we may take \( \rho = e^{-r} \). If the 0-volume of the funnels is zero, then the 0-volume of \( X \) equals the volume of \( \hat{X}_\tau \), which is \(-2\pi \chi(X) \) by Gauss-Bonnet. We will use 0-tr\( \tau \) to denote the 0-trace with respect to a defining function in this class. This is the same convention used for the 0-trace in [17, 33].

**Theorem 3.2.** Suppose that \( \tau \) is a convex co-compact hyperbolic metric on \( X \), and let \( D_\tau(s) \) be any function defined by (2.2) using 0-tr\( \tau \). Then \( D_\tau(s) \) is an entire function of order two with zeros \( \zeta \in R_\tau \) of multiplicity \( m_\zeta \), given by the formula
\[
D_\tau(s) = e^{F s(s-1)} + G Z_\tau(s) Z_\infty(s)
\]
where \( F \) and \( G \) are the free parameters in the definition. Therefore \( D_\tau(s) \) is determined (up to the constants \( F \) and \( G \)) by the set \( R_\tau \) of eigenvalues and resonances of \( \Delta_\tau \) counted with multiplicities.

**Remarks.**

1. Sarnak [35] showed that the (zeta-regularized) determinant of the Laplacian on a compact hyperbolic surface \( S \) is given by
\[
det(\Delta_S + (s - 1)) = Z_S(s) \left( e^{E - s(s-1)/2} \frac{\Gamma_2(s)^2}{\Gamma(s)} \right)^{-\chi(S)}
\]
where \( Z_S(s) \) is the zeta function for the compact surface, and
\[
E = -\frac{1}{4} - \frac{1}{2} \log(2\pi) + 2\zeta'(-1),
\]
with \( \zeta(s) \) the Riemann zeta function. In particular,
\[
det(\Delta_S) = c(\chi(S)) Z_S(1)
\]
for a constant depending only on the Euler characteristic. Thus our determinant has the same form as Sarnak’s; from this point of view a natural choice for the constants \( F \) and \( G \) in (3.8) would be \( F = \chi(X) \) and \( G = -\chi(X)E \).
2. Taking $s = 1$ gives a determinant of the Laplacian in terms of special values of the zeta function (in contrast to \[35\] there is no derivative because neither the zeta function nor the determinant has a zero at $s = 1$): one has

$$\det \Delta_g = e^{G(2\pi) - \chi(X)} Z_\tau(1).$$

**Proof.** To prove Theorem 3.2, we study the function $L(s) = \frac{d}{ds} \log D_\tau(s)$, defined up to one parameter by the relation

$$\frac{1}{2s - 1} \frac{d}{ds} \left( \frac{1}{2s - 1} L(s) \right) = -0\text{-tr}_\tau R_\tau(s)^2.$$  

We recall from [33] the relation

$$\frac{1}{2s - 1} \frac{d}{ds} \log Z_\tau(s) = 0\text{-tr}_\tau (R_\tau(s) - R_{\text{H}}(s)),$$

where $R_{\text{H}}(s)$ is the resolvent of the Laplacian on the two-dimensional hyperbolic space $\mathbb{H}$, and the integral kernel on the right is obtained by lifting the integral kernel of $R_\tau(s)$ to $\mathbb{H} \times \mathbb{H}$, subtracting that of $R_{\text{H}}(s)$, and restricting to the diagonal; the resulting restriction is smooth and projects to a function on $\overline{X}$ (see [33], Section 6). The 0-trace in (3.10) refers, by a slight abuse of notation, to the 0-integral of this smooth function. A further differentiation gives

$$\left( \frac{1}{2s - 1} \frac{d}{ds} \log Z_\tau(s) \right)^2 = \frac{1}{2s - 1} \frac{d}{ds} \left( \frac{1}{2s - 1} L(s) \right) + 0\text{-tr}_\tau (R_{\text{H}}(s)^2)$$

where the second right-hand term is interpreted as follows. The integral kernel of $R_{\text{H}}(s)^2$ is continuous and its restriction to the diagonal of $\mathbb{H} \times \mathbb{H}$ equals the constant

$$-\frac{1}{2\pi} \frac{1}{2s - 1} \frac{d}{ds} \psi(s),$$

where $\psi(s)$ is the logarithmic derivative of $\Gamma(s)$. Recall for later use that

$$\psi(s) = -\frac{1}{s} - \gamma + \sum_{k=1}^{\infty} \frac{s}{k(s+k)}.$$  

As noted above, $0\text{-vol}_\tau(X) = -2\pi \chi(X)$, so we have

$$0\text{-tr}_\tau (R_{\text{H}}(s)^2) = \frac{\chi(X)}{2s - 1} \frac{d}{ds} \psi(s).$$

From (3.11) we thus see that

$$L(s) = \frac{d}{ds} \log Z_\tau(s) + \frac{\chi(X)(2s - 1)}{2s - 1} \psi(s) + C(2s - 1)$$

for some constant $C$.

We now compare (3.3) and (3.12) to conclude that

$$\psi(s) = \frac{1}{\chi(X)} \frac{1}{2s - 1} \frac{d}{ds} \log Z_\infty(s) + 1$$

so that (3.13) becomes

$$L(s) = \frac{d}{ds} \log(Z_\tau(s) Z_\infty(s)) + \left(\chi(X) + C(2s - 1)\right).$$

Setting $F = C + \chi(X)$, we obtain the formula (3.8). \qed
4. Relative determinant

4.1. Conformal deformation theory. On a compact surface any metric is conformal related to unique hyperbolic metric. This is the basis for the isospectral result of Osgood-Phillips-Sarnak \[27\]. In the non-compact case, Mazzeo-Taylor recently proved a uniformization result well-suited to our situation.

**Theorem 4.1.** \[27\] Let \( h \) be an arbitrary smooth metric on \( \bar{\mathbb{X}} \). Then there exists a function \( u \in \rho C^\infty(\bar{\mathbb{X}}) \) such that \( e^{2u}\rho^{-2}h \) is a complete hyperbolic metric on \( \mathbb{X} \).

(A theorem of Hulin-Troyanov \[19\] gives a similar result in this situation with \( u \) bounded and smooth on \( \mathbb{X} \), but with no boundary regularity on \( \bar{\mathbb{X}} \).) If \( g \) denotes the conformally compact metric \( \rho^{-2}h \), then \( u \) is a solution of the equation

\[
\Delta g u = -K(g) - e^{2u}.
\]

By examining this closely we can gain an extra order of vanishing of \( u \) with a curvature restriction.

**Corollary 4.2.** Let \( g \) be a conformally compact metric on \( \mathbb{X} \). If \( K(g) + 1 = \mathcal{O}(\rho^2) \), then there exists a function \( u \in \rho^2 C^\infty(\bar{\mathbb{X}}) \) such that \( e^{2u}g \) is hyperbolic on \( \mathbb{X} \).

**Proof.** The theorem gives \( u \in \rho C^\infty(\bar{\mathbb{X}}) \) satisfying (4.1), which we rewrite as

\[
(\Delta_g + 2) u = -(1 + K(g)) + 1 + 2u - e^{2u}.
\]

The first term on the right-hand side is \( \mathcal{O}(\rho^2) \) by assumption, and the remainder is also because of the vanishing of \( u \) to first order. Thus \( (\Delta_g + 2) u = \mathcal{O}(\rho^2) \). Now if we write \( u = \rho f \), then because \( g \) is an asymptotically hyperbolic metric on \( \mathbb{X} \), a straightforward computation shows \((\Delta_g + 2)(\rho f) = 2\rho f + \mathcal{O}(\rho^2)\). Thus \( f \) vanishes at \( \rho = 0 \) and \( u \) is in fact in \( \rho^2 C^\infty(\bar{\mathbb{X}}) \).

Note that the curvature condition is stronger than the asymptotically hyperbolic assumption, which implies only that \( K(g) + 1 = \mathcal{O}(\rho) \).

4.2. Zeta-regularization of the relative determinant. Let \( g \) be an asymptotically hyperbolic metric on \( \mathbb{X} \) such that \( K(g) + 1 = \mathcal{O}(\rho^2) \). According to Corollary \[12\] this can be written as \( g = e^{2\varphi} \tau \) where \( \varphi \in \rho^2 C^\infty(\bar{\mathbb{X}}) \) and \( \tau \) is convex co-compact hyperbolic. To define a relative determinant from \( \Delta_g \) to \( \Delta_\tau \) we need first to compare operators acting on the same space. Note that \( \Delta_g = e^{-2\varphi} \Delta_\tau \) and \( dg = e^{2\varphi} d\tau \). Thus

\[
U f = e^{\varphi} f
\]

is a unitary transformation from \( L^2(\mathbb{X}, dg) \) to \( L^2(\mathbb{X}, d\tau) \), and the pull-back of \( \Delta_g \) under this map is

\[
\hat{\Delta}_g = U \Delta_g U^{-1} = e^{-\varphi} \Delta_\tau e^{-\varphi}.
\]

We will define the relative determinant through a relative zeta function, as in \[30\]. For this to be well-defined we need \( e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau} \) to be trace-class for \( t > 0 \). For \( \varphi \) compactly supported this can be seen immediately from Duhamel’s formula:

\[
e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau} = \int_0^t e^{-s\Delta_\tau} \left( \Delta_\tau - \hat{\Delta}_g \right) e^{-(t-s)\Delta_g} ds.
\]

For the general case, we will prove a slightly stronger condition. Let \( R_\tau(s) \) be the resolvent for \( \Delta_\tau \), and define

\[
\hat{R}_g(s) = (\hat{\Delta}_g + s(s - 1))^{-1} = e^{\varphi} R_\varphi(s) e^{-\varphi}.
\]
Note that in the definition (2.2) of \( D_g(s) \), replacing \( R_g(s) \) by \( \hat{R}_g(s) \) has no effect.

**Lemma 4.3.** Let \( g \) be an asymptotically hyperbolic metric with \( K(g) + 1 = O(\rho^2) \). For \( \text{Re}(s) > 1 \) the operator \( \hat{R}_g(s)^2 - R_\tau(s)^2 \) is trace-class.

**Proof.** A detailed description of the structure of the resolvent \( R_g(s) \) was obtained in [20]. From this picture one can easily deduce that the operator \( \rho \) is Hilbert-Schmidt provided that \( \text{Re}(s) > 1/2 - \min\{\alpha, \beta\} \) and \( \alpha + \beta > 1/2 \). The same fact holds for \( R_\tau(s) \).

We write
\[
\hat{R}_g(s)^2 - R_\tau(s)^2 = \hat{R}_g(s)(\hat{R}_g(s) - R_\tau(s)) + (\hat{R}_g(s) - R_\tau(s))R_\tau(s)
\]
(4.2)
\[
= \hat{R}_g(s)^2(\Delta_\tau - \hat{\Delta}_g)R_\tau(s) + \hat{R}_g(s)(\Delta_\tau - \hat{\Delta}_g)R_\tau(s)^2
\]
We will show that the first of these terms is a trace-class operator since the analysis of the second term is similar. Expanding
\[
\Delta_\tau - \hat{\Delta}_g = (1 - e^{-\varphi})\Delta_\tau e^{-\varphi} + \Delta_\tau (1 - e^{-\varphi}),
\]
we can re-express the right-hand first term in (4.2) as
\[
\hat{R}_g(s)^2(e^{-\varphi} - 1)\Delta_\tau e^{-\varphi}R_\tau(s) + \hat{R}_g(s)^2\Delta_\tau (e^{-\varphi} - 1)R_\tau(s).
\]
Once again, the two terms are very similar and we will consider only the first. Since \( \Delta_\tau e^{-\varphi}R_\tau(s) \) is a bounded operator for \( \text{Re}(s) > 1/2 \), we may ignore this part. The function \((e^{-\varphi} - 1) = \rho^2 h \) for \( h \in C^\infty(\overline{X}) \), so the analysis reduces to considering \( \hat{R}_g(s)^2\rho^2 \). Writing this as \( \hat{R}_g(s)^2\rho^2 \), we see from the characterization of Hilbert-Schmidt operators above that this term is trace class for \( \text{Re}(s) > 1 \).  

By the Birman-Krein spectral shift theory (which we will use in more detail below), we deduce the following:

**Corollary 4.4.** For \( g \) an asymptotically hyperbolic metric with \( K(g) + 1 = O(\rho^2) \), the relative heat-kernel \( e^{-t\Delta_g} - e^{-t\Delta_\tau} \) is trace-class for \( t > 0 \).

As \( t \to 0 \), standard heat kernel asymptotics can be used to derive an expansion
\[
\text{tr } [e^{-t\Delta_g} - e^{-t\Delta_\tau}] \sim \frac{1}{t} \sum_{j \geq 0} a_j t^j.
\]
(4.3)
Let \( \mathcal{H}_g(t, x, y) \) be the heat kernel for the metric \( g \), i.e. the Schwarz kernel of \( e^{-t\Delta_g} \) with respect to the Riemannian measure \( dg \). If \( \hat{\mathcal{H}}_g(t, x, y) \) denotes the Schwarz kernel of \( e^{-t\Delta_\tau} \) with respect to \( d\tau \), then the relationship between these kernels is
\[
\hat{\mathcal{H}}_g(t, x, y) = e^{\varphi(x)}\mathcal{H}_g(t, x, y)e^{\varphi(y)}.
\]
Thus, the local heat expansion for \( \Delta_g \) is easily related to the standard results for \( \Delta_\tau \) (see [28]). The zeroth relative heat invariant measures the change in area:
\[
a_0 = \frac{1}{4\pi} \int_X (e^{2\varphi} - 1) d\tau.
\]
(4.4)
The formula for the first invariant is
\[
a_1 = \frac{1}{12\pi} \int_X (e^{2\varphi} K_g - K_\tau) d\tau,
\]
(4.5)
where the $K$'s are the respective Gaussian curvatures. Since $K_\tau = -1$ and $K_g = e^{-2\varphi}(\Delta_\tau \varphi - 1)$, we have

$$a_1 = \frac{1}{12\pi} \int_X \Delta_\tau \varphi \, d\tau = 0.$$  

(The first heat invariant in the compact case is the Euler characteristic; here the first relative heat invariant is zero because the Euler characteristic for $\tau$ to $g$ are the same.) The second heat invariant involves the square of the curvature:

$$a_2 = \frac{1}{60\pi} \int_X (e^{2\varphi} K_g^2 - 1) \, d\tau,$$

and the higher invariants are integrals of polynomials in $K_g$ and $\Delta_g$:

$$a_j = c_j \int_X e^{2\varphi} K_g \Delta_g^{j-2} K_g \, d\tau + (\text{terms with fewer derivatives})$$

where $c_j \neq 0$.

Let $\lambda_g = \inf \text{spec}(\hat{\Delta}_g)$ and $\lambda_\tau = \inf \text{spec}(\Delta_\tau)$. Both of these numbers are strictly positive since $X$ does not support constant eigenfunctions. If $\mu = \min(\lambda_g, \lambda_\tau)$, then we can deduce that

$$\text{tr} \left[ e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau} \right] = O(e^{-\mu t})$$

as $t \to \infty$.

The relative zeta function is initially defined for $\Re(w) > 1$ and $\Re(s(s-1)) > -\mu$ by

$$\zeta(w, s) = \frac{1}{\Gamma(w)} \int_0^\infty t^w e^{-ts(s-1)} \text{tr} \left[ e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau} \right] \, \frac{dt}{t}$$

We wish to define the relative determinant by

$$\log D_{g,\tau}(s) = -\zeta_w(0, s),$$

where the subscript denotes differentiation with respect to the $w$ variable. To justify this definition, note that the heat expansion (4.3) implies that

$$\zeta(w, s) = a_0 \frac{s(s-1)^{w-1}}{w-1} + (\text{analytic for } \Re(w) > -1),$$

so (4.10) is well-defined by meromorphic continuation in $w$, at least for $\Re(s(s-1)) > -\mu$.

The next result will enable us to connect the relative and absolute determinants.

**Lemma 4.5.** The identity

$$\left( \frac{1}{2s-1} \frac{d}{ds} \right)^2 \log D_{g,\tau}(s) = -\text{tr} \left[ \hat{R}_g(s)^2 - R_\tau(s)^2 \right]$$

holds for $\Re(s) > 1$.

**Proof.** For $\Re[s(s-1)] > -\mu$, the function $\zeta(w, s)$ is continuously differentiable to all orders in $s$ and $w$ near $w = 0$ so that we may calculate

$$\left( \frac{1}{2s-1} \frac{d}{ds} \right)^2 \log D_{g,\tau}(s) = \frac{d}{dw} \left. \left( -w(w+1)\zeta(w+2, s) \right) \right|_{w=0} = -\zeta(2, s).$$
On the other hand, $\hat{R}_g(s)^2 - R_\tau(s)^2$ is trace-class for $\text{Re}(s) > 1$ by Lemma 4.3. For real $s$ with $s > 1$ and real $w$ with $w \geq 2$ we can therefore identify

$$\zeta(w, s) = \text{tr} \left[ \hat{R}_g(s)^w - R_\tau(s)^w \right]$$

(4.12)

(the power is well-defined since the resolvents are positive operators if $s > 1$). The result now follows for all real $s > 1$ and hence all $s$ with $\text{Re}(s) > 1$ by analytic continuation.

In order to get the correct relation to the zeta function, in §3 we defined $D_\tau(s)$ using $0\text{-tr}_\tau$, the 0-trace for a class of defining function canonically associated to $\tau$. Now, in order to get the proper connection to the relative determinant, we must also use $0\text{-tr}_\tau$ to define $D_g(s)$. Then, since

$$\text{tr} \left[ \hat{R}_g(s)^2 - R_\tau(s)^2 \right] = 0\text{-tr}_\tau \left[ \hat{R}_g(s)^2 \right] - 0\text{-tr}_\tau \left[ R_\tau(s)^2 \right]$$

for $\text{Re}(s) > 1$, it follows that

$$\left( \frac{1}{2s - 1} \frac{d}{ds} \right)^2 \log D(s) = \left( \frac{1}{2s - 1} \frac{d}{ds} \right)^2 \log(D_{g,\tau}(s)D_\tau(s)).$$

Thus:

**Lemma 4.6.** Let $D_\tau(s)$ and $D_g(s)$ be defined by (2.2) using $0\text{-tr}_\tau$. Then there are constants $E$ and $F$ so that

$$D_g(s) = e^{E + Fs(s-1)}D_{g,\tau}(s)D_\tau(s).$$

In particular, if $g$ is hyperbolic near infinity then $D_{g,\tau}(s)$ admits a meromorphic continuation to the whole plane.

To conclude this subsection, we note that the Birman-Krein theory of the spectral shift (see e.g. [35], chapter 8 for an exposition) applies to give a measurable, locally integrable function $\xi$ on $[0, \infty)$, the spectral shift function, with the property that

$$\int \phi'(\lambda)d\xi(\lambda) = \text{tr}(\phi(\hat{\Delta}_g) - \phi(\Delta_\tau))$$

for any smooth function $\phi$ vanishing rapidly at infinity. In particular we have

$$\text{tr} \left[ e^{-t\hat{\Delta}_g} - e^{-t\Delta_\tau} \right] = -t \int_0^\infty e^{-t\lambda} \xi(\lambda) \, d\lambda.$$
4.3. Polyakov formula. In this subsection we will establish the Polyakov formula stated in the introduction.

Proof of Proposition 1.2. Consider the variation \( \varphi \to \varphi + \delta \varphi \) for some function \( \delta \varphi \in \rho^2C^\infty(\bar{X}) \). Since \( \hat{\Delta}_g = e^{-\varphi} \Delta_g e^{-\varphi} \) we have

\[
\delta \hat{\Delta}_g = -\delta \varphi \hat{\Delta}_g - \hat{\Delta}_g \delta \varphi.
\]

Because of the decay of \( \delta \varphi \) at the boundary, we can differentiate inside the trace to obtain

\[
\delta \text{tr} \left[ e^{-t \hat{\Delta}_s} - e^{-t \hat{\Delta}_s} \right] = t \text{tr} \left[ \delta \varphi \hat{\Delta}_g e^{-t \hat{\Delta}_s} + \hat{\Delta}_g \delta \varphi e^{-t \hat{\Delta}_s} \right]
= 2 t \text{tr} \left[ \delta \varphi \hat{\Delta}_g e^{-t \hat{\Delta}_s} \right].
\]

Then the variation of the relative zeta function (at \( s = 1 \)) is

\[
\delta \zeta(w, 1) = 2 \Gamma(w) \int_0^\infty t^w \text{tr} \left[ \delta \varphi \hat{\Delta}_g e^{-t \hat{\Delta}_s} \right] dt
= - \frac{2}{\Gamma(w)} \int_0^\infty t^w \frac{d}{dt} \text{tr} \left[ \delta \varphi e^{-t \hat{\Delta}_s} \right] dt
= \frac{2w}{\Gamma(w)} \int_0^\infty t^{w-1} \text{tr} \left[ \delta \varphi e^{-t \hat{\Delta}_s} \right] dt.
\]

Now from the expansion of the heat kernel at \( t = 0 \) we have

\[
\text{tr} \left[ \delta \varphi e^{-t \hat{\Delta}_s} \right] = b_0 t^{-1} + b_1 + O(t),
\]

where

\[
b_1 = \frac{1}{12\pi} \int_X \delta \varphi (\Delta_\tau \varphi - 1) d\tau.
\]

The heat expansion implies that

\[
\delta \zeta(w, 1) = \frac{2w}{\Gamma(w)} \left[ \frac{b_0}{w-1} + \frac{b_1}{w} + \text{(analytic for } \text{Re}(w) > -1) \right],
\]

from which we directly obtain

\[
\delta \log D_{g,\tau}(1) = -2b_1 = -\frac{1}{6\pi} \int_X \delta \varphi (\Delta_\tau \varphi - 1) d\tau.
\]

Integrating this expression gives the Polyakov formula.

5. Hadamard factorization of the relative determinant

For this section and the remainder of the paper we will restrict our attention to the case of a metric \( g \) which is hyperbolic near infinity, so that \( D_{g,\tau}(s) \) is a meromorphic function by Lemma 4.6.

Let us introduce the Hadamard product

\[
P_g(s) = \prod_{\zeta \in \mathcal{R}_g} \left( 1 - \frac{s}{\zeta} \right)^{m_{\zeta}} e^{m_{\zeta}(-\frac{s}{\zeta} + \frac{\tau}{s})},
\]

whose zeroes are given by \( \mathcal{R}_g \). Convergence of the Hadamard product is guaranteed by the estimate

\[
\# \{ \zeta \in \mathcal{R}_* : |\zeta| \leq r \} \leq C(1 + r^2)
\]

proven in [15].
The goal of this section is the proof of Theorem 1.1. This will be divided into three steps: first, we show that the divisors of $D_{g,\tau}(s)$ and the quotient $P_g(s)/P_\tau(s)$ coincide, so that the two differ by an holomorphic function without zeros. Next, we show that the logarithmic derivative of

$$W(s) = D_{g,\tau}(s) \frac{P_\tau(s)}{P_g(s)}$$

has at most polynomial growth. Finally, we obtain an exact growth rate by examining the asymptotic behavior of $\log D_{g,\tau}(s)$ as $s \to \infty$ through real values.

### 5.1. Divisor of the relative determinant

In this subsection, we prove:

**Proposition 5.1.** For $g$ hyperbolic near infinity, the relative determinant $D_{g,\tau}(s)$ has a meromorphic extension to $C$, with divisor equal to that of $P_g(s)/P_\tau(s)$.

The connection between $D_{g,\tau}(s)$ and the resonances will be made through scattering theory. Let $S_g(s)$ and $S_\tau(s)$ be respective scattering operators (on $\partial X$) for $\Delta_g$ and $\Delta_\tau$ as defined in [17], Definition 2.12. By [21], Theorem 7.1 we have that $S_g(s) - S_\tau(s)$ is a pseudodifferential operator of order $2 \text{Re}(s) - 3$. Thus the relative scattering operator $S_{g,\tau}(s) = S_g(s)S_\tau(s)^{-1}$ satisfies $S_{g,\tau}(s) = I + Q(s)$, where $Q(s)$ is of order $-2$ and hence trace class.

Our first result follows from the analysis in Proposition 2.5 of [17].

**Lemma 5.2.** The determinant of the relative scattering operator satisfies

$$\det S_{g,\tau}(s) = e^{f_1(s)} \frac{P_g(1-s)}{P_g(s)} \frac{P_\tau(s)}{P_\tau(1-s)},$$

for some polynomial $f_1$ of degree at most four.

Since both $P_g(1-s)$ and $P_g(s)$ appear in Lemma 5.2, in order to make use of it we must first explicitly count poles of $D_{g,\tau}(s)$ in the half-plane $\text{Re}(s) \geq 1/2$.

**Lemma 5.3.** The divisor of $D_{g,\tau}(s)$ in the closed half-plane $\text{Re}(s) \geq 1/2$ coincides with that of $P_g(s)/P_\tau(s)$.

**Proof.** To determine the zeros and poles of $D_{g,\tau}(s)$ in the closed half-plane $\text{Re}(s) \geq 1/2$, we want to consider the trace of $\hat{R}_g(s) - R_\tau(s)$. This does not even have a 0-trace, so instead we fix $s_0$ not a pole of either operator and examine

$$H(s) = (2s - 1) \text{0-tr}_\tau \left[ \hat{R}_g(s) - R_\tau(s) - \hat{R}_g(s_0) + R_\tau(s_0) \right].$$

By Lemma 5.2 we have

$$\frac{1}{2s - 1} \frac{d}{ds} \left( \frac{1}{2s - 1} H(s) \right) = \left( \frac{1}{2s - 1} \frac{d}{ds} \right)^2 \log D_{g,\tau}(s).$$

Integrating gives

$$\frac{d}{ds} \log D_{g,\tau}(s) = H(s) + c(2s - 1)$$

for some constant $c$. For $* = g$ or $\tau$, let $R_*(s)$ denote $\hat{R}_g(s)$ or $R_\tau(s)$, respectively. Let $m_\zeta^*$ be the multiplicity of the pole of $R_*(s)$ at $\zeta$, with $m_\zeta^* = 0$ if there is no pole. For $\text{Re}(\zeta) > \frac{1}{2}$, the only poles occur at real $\zeta$ and correspond to eigenvalues. At such a value the pole of $R_*(s)$ is simple, and the residue of $(2s - 1)R_*(s)$ is a rank-$m_\zeta^*$ projection operator. From this it follows that for $\text{Re}(\zeta) > 1/2$ $H(s)$ has
a simple pole with residue exactly $m_1^2 - m_2^2$. Then by (3.4) $D_{g,\tau}(s)$ has divisor
$
\{\zeta, m_1^2 - m_2^2\}$ for $\Re(\zeta) > 1/2$, which is the divisor of $P_g(s)/P_\tau(s)$.

The only singularity of $\hat{R}_g(s)$ or $R_\tau(s)$ that can occur on the line $\Re(s) = 1/2$

is at $s = 1/2$ (see [17], Lemma 4.1). It follows from that Lemma and the fact that

there are no $L^2(X)$-eigenfunctions of $\Delta_g$ or $\Delta_\tau$ with eigenvalue $1/4$ that

$$R_\tau(s) = \frac{B_\tau(s)}{2s-1} + C_\tau(s),$$

where $B_\tau(s)$ and $C_\tau(s)$ are analytic at $s = 1/2$ and $B_\tau(s)$ is a finite-rank operator.

Although $(2s-1)R_\tau(s)$ is thus analytic near $s = 1/2$, the definition of the zero-trace

together with the spatial decay of the kernels may introduce a pole there, as may be

se 4 seen by considering the small-$\varepsilon$ expansion of the model integral $\int_\varepsilon y^{2s-2}dy$. From

[33], Lemma 4.9 we obtain that

$$B_\tau(s; z, z') = \sum_{k=1}^{N_\varepsilon} u_k^*(s; z)v_k^*(s; z'),$$

with $u_k^*(s), v_k^*(s) \in \rho^*C^\infty(\bar{X})$. These functions have the property that the restrictions

$\tilde{\bar{u}}_k = (\rho^{-1/2}u_k^*(1/2))|_{\partial\bar{X}}$ and $\tilde{\bar{v}}_k = (\rho^{-1/2}v_k^*(1/2))|_{\partial\bar{X}}$ are both non-zero.

Analyzing as in the proof of Theorem 6.2 of [33], we see that $0\text{-tr}_\tau B_\tau(s)$ has a

simple pole at $s = 1/2$ with residue given by

$$-\frac{1}{2} \int_{\partial\bar{X}} \tilde{\bar{u}}_k\tilde{\bar{v}}_k^* dh\tau,$$

where $h\tau = (\rho^2 g)|_{\partial\bar{X}}$ and similarly for $h_\tau$. By Lemma 4.16 of [33] this is equal to

$m_{1/2}^1$. Thus the residue of $H(s)$ at $s = 1/2$ is $m_{1/2}^2 - m_{1/2}^1$, which gives the proper

singularity for $D_{g,\tau}(s)$.

The final step is the connection between $D_{g,\tau}(s)$ and $\det S_{g,\tau}(s)$.

**Lemma 5.4.**

$$\det S_{g,\tau}(s) = e^{f_2(s)} \frac{D_{g,\tau}(1-s)}{D_{g,\tau}(s)},$$

for a polynomial $f_2$ of degree at most two.

**Proof.** We will study the function

$$F(s) = (2s-1)0\text{-tr}_\tau[\hat{R}_g(s) - \hat{R}_g(1-s) - R_0(s) + R_0(1-s)].$$

By [33] we see that

$$\frac{d}{ds} \log[D_{g,\tau}(s)/D_{g,\tau}(1-s)] = F(s) + 2c(2s-1)$$

We claim that

$$\frac{d}{ds} \log(\det S_{g,\tau}(s)) = -F(s),$$

which will complete the proof. The formula (5.6) is established by a rather long

technical calculation, which we defer to Appendix [33].

Combining Lemma 5.2 and Lemma 5.4 gives the functional equation

$$\frac{D_{g,\tau}(1-s)}{D_{g,\tau}(s)} = e^{f(s)} \frac{P_g(s)}{P_g(1-s)} \frac{P_\tau(s)}{P_\tau(1-s)},$$
where \( f \) is a polynomial of degree at most four. In conjunction with Lemma 5.3, this proves Proposition 5.1.

5.2. Growth estimates on the relative determinant. The results of the preceding subsection show that \( W(s) = D_{g,\tau}(s)P_\tau(s)/P_g(s) \) is entire and zero-free. We now wish to estimate the growth of \( \log W(s) \) as \( |s| \to \infty \) and show it to be polynomial. Using (5.3) and Lemma 4.5, it is easy to see that

\[
\frac{d}{ds} \log W(s) = H(s) + c(2s - 1) + P'_\tau(s)/P_\tau(s) - P'(s)/P_g(s).
\]

for some constant \( c \).

It will suffice to show that the right-hand side of (5.8) has polynomial growth as \( |s| \to \infty \). By the estimate (5.2), for any \( \delta > 0 \) there is a countable collection of disjoint discs \( \{D_j\} \) with the properties that

1. \( R_g \cup R_\tau \subset \bigcup_j D_j \)
2. \( \text{dist}(s, R_g \cup R_\tau) \geq C \langle s \rangle^{-2-\delta} \) for every \( s \in \mathbb{C} \setminus (R_g \cup R_\tau) \).

Here \( \langle s \rangle = (1 + |s|^2)^{1/2} \). To prove that \( W'(s)/W(s) \) has polynomial growth, it suffices by the maximum modulus theorem to prove a polynomial growth estimate in \( \mathbb{C} \setminus (\bigcup_j D_j) \). Standard estimates on Hadamard products show that, on \( \mathbb{C} \setminus (\bigcup_j D_j) \), the estimates

\[
|P'_\tau(s)/P_\tau(s)| \leq C\langle s \rangle^{4+\delta}
\]

and

\[
|P'(s)/P_g(s)| \leq C\langle s \rangle^{4+\delta}
\]

hold. Thus we need only show that the first right-hand term in (5.8), which is \( \frac{d}{ds} \log D_{g,\tau}(s) \) up to a linear polynomial, grows polynomially in \( \mathbb{C} \setminus (\bigcup_j D_j) \). By the functional equation (5.7) and these observations, it suffices to show that, for some \( \epsilon > 0 \),

\[
\frac{d}{ds} \log D_{g,\tau}(s) \text{ grows polynomially in the half-plane } \text{Re}(s) > \frac{1}{2} - \epsilon \text{ with the discs } D_j \text{ removed.}
\]

To this end, we first examine the behavior of

\[
\left( \frac{1}{2s-1} \frac{d}{ds} \right)^2 \log D_{g,\tau}(s) = \text{tr} \left[ \hat{R}_g(s)^2 - R_\tau(s)^2 \right]
\]

using the spectral shift representation.

Lemma 5.5. Let \( \epsilon > 0 \). For \( \text{Re}(s) > \frac{1}{2} + \epsilon \) and \( |s| > 1 \) we can estimate

\[
\left| \text{tr} \left[ \hat{R}_g(s)^2 - R_\tau(s)^2 \right] \right| \leq C \epsilon.
\]

Proof. By (4.12) and (4.15) we can express the trace we are trying to estimate in terms of the spectral shift function:

\[
\text{tr} \left[ \hat{R}_g(s)^2 - R_\tau(s)^2 \right] = -2 \int_{\mu}^{\infty} (\lambda + s(s - 1))^{-3} \xi(\lambda) d\lambda.
\]

The integrand can be factored as

\[
(\lambda + s(s - 1))^{-3} = \lambda^{-3} [1 + s(s - 1)/\lambda]^{-3}
\]
Since the assumptions on $s$ keep $s(s - 1)$ bounded away from $(-\infty, -\mu]$, we can estimate
\[ |1 + s(s - 1)/\lambda|^{-3} < C, \]
uniformly for $\lambda \in [\mu, \infty)$ and $s$ as indicated. The result then follows from (1.14).

Integration of (5.9) now yields the estimate
\[ |\log D_{g,\tau}(s)| \leq C\langle s \rangle^4 + \varepsilon \]
for $\text{Re}(s) > 1/2 + \varepsilon$ and $|s| > 1$. Let $S_\varepsilon$ be the open strip
\[ S_\varepsilon = \{1/2 - \varepsilon < \text{Re}(s) < 1/2 + \varepsilon\}. \]
Using the functional equation (5.7) and the estimates on Hadamard products, we can now bound
\[ |\log H(s)| \leq C\langle s \rangle^{4+\varepsilon} \]
for $S \in \mathbb{C}\setminus S_\varepsilon$.

To complete the proof we must estimate $|\log D_{g,\tau}(s)|$ in $S_\varepsilon$, which is rather delicate because of the poles. Fortunately, by the Phragmén-Lindelöf Theorem an exponential growth estimate in the strip will suffice to extend the polynomial bounds.

**Lemma 5.6.** The estimate
\[ |\log D_{g,\tau}(s)| \leq C(\eta) \exp(|s|^{2+\eta}) \]
holds for any $\eta > 0$ and all $s$ with $s \in S_\varepsilon \setminus \bigcup_j D_j$.

The proof is quite technical and relies on a parametrix construction and estimates from Guillopé-Zworski [17]. We will review this construction and prove Lemma 5.6 in Appendix B.

Since $P_g(s)$ and $P_\tau(s)$ are of finite order, and the Maximum Modulus Theorem can be used to fill in estimates in the disks $D_j$, Lemma 5.6 gives us an estimate
\[ \log |W(s)| \leq C \exp(|s|^{2+\eta}) \]
for $s \in S_\varepsilon$. In $\mathbb{C}\setminus S_\varepsilon$, we have
\[ |\log W(s)| \leq C\langle s \rangle^{4+\delta}. \]
The Phragmén-Lindelöf Theorem now applies to give us a polynomial bound on $\log W(s)$ over the whole plane. We have shown:

**Proposition 5.7.** $W(s) = e^{q(s)}$ where $q(s)$ is a polynomial of degree at most 4.

5.3. **Asymptotics and order.** To show that $q(s)$ has order at most two, and that it depends only on the eigenvalues and resonances of $\Delta_g$, let us rewrite the equation in Theorem 1.1 as
\[(5.10) \quad \log D_{g,\tau}(s) = q(s) + \log P_g(s) - \log Z_\tau(s) + \log Z_\infty(s),\]
and consider the asymptotic behavior of each term as $\text{Re}(s) \to \infty$. By Proposition 5.7, $q(s)$ is polynomial of degree at most 4. The definition of $Z_\tau(s)$ in terms of the length spectrum (see (3.1)) together with the crude estimate that
\[ \#\{\gamma : l(\gamma) \leq t\} \leq C e^t \]
shows that

\[ |\log Z_\tau(s)| \leq Ce^{-c\text{Re}(s)} \]

for \(\text{Re}(s) > 1\). This fact is rather crucial, since it shows that the resonances of \(\tau\) have no influence on the asymptotic expansion.

The expansion of \(\log Z_\infty(s)\) has been given in \([3.6]\). The asymptotics of the relative determinant can be obtained via the heat expansion. This analysis is identical to the expansion of the zeta regularized determinant in the compact case as given in \([35]\), with the important exception that the first relative heat invariant \(a_1 = 0\). For \(\text{Re}(s) \to \infty\),

\[
\log D_{g,\tau}(s) \sim a_0 s(s - 1)[1 - \log s(s - 1)] + \sum_{j=2}^{\infty} (j - 2)! a_j z^{-j+1}. 
\]

The following result will be critical to the consideration of isopolar sequences of metrics, because (in contrast to the compact case) the relative determinant \(D_{g,\tau}(s)\) is not a spectral invariant.

**Proposition 5.8.** The Euler characteristic \(\chi(X)\), the polynomial \(q(s)\), and all of the relative heat invariants \(a_j\) are determined by the set of eigenvalues and resonances of \(\Delta_g\).

**Proof.** The eigenvalues and resonances of \(\Delta_g\) fix the Hadamard factor \(P_g(s)\), which must have an asymptotic expansion for \(\text{Re}(s) \to \infty\) by the analysis of the other terms in \([5.10]\). Since \(a_1 = 0\), the only term of the form \(\log s\) in the expansions is \(\frac{1}{3}\chi(X)\log s\). Therefore this term must be cancelled by a corresponding term in the expansion of \(\log P_g(s)\), from which we see that \(\chi(X)\) is an isopolar invariant (which implies \(Z_\infty(s)\) is also). Knowing this, we observe that none of the terms in the expansions of \(q(s)\) and \(\log D_{g,\tau}(s)\) could cancel with each other and conclude that all coefficients are isopolar invariants.

To complete the proof of Theorem 1.1 we note that \(\log P_g(s) = O(|s|^2 \ln |s|)\) as \(s \to \infty\) with a similar estimate for \(P_\tau(s)\). On the other hand the heat expansion \([5.11]\) can be used to derive an \(O(|s|^2 \ln |s|)\) estimate for \(\log D_{g,\tau}(s)\) in a sector such as \(|\arg(s)| \leq \varepsilon\). It follows that the degree of \(q(s)\) is actually two or less.

6. **Compactness for isopolar classes**

Let \(\{g_k\}\) be a sequence of isopolar metrics on \(X\) satisfying the hypotheses of Theorem 1.4. By pulling back by diffeomorphisms if necessary, we will assume that the uniformizing metrics \(\tau_k\) are arranged so that their convex cores coincide. That is, \(\hat{X}_{\tau_k} = K\) for some fixed compact set \(K \subset X\). (We will see below that such an alignment of the convex cores is necessary for convergence of a subsequence.) Then by assumption we have \(\text{supp}(\varphi_k) \subset K\). We wish to establish the existence of a convergent subsequence in the \(\rho^{-2}C^\infty(\hat{X})\) topology.

The Hadamard factor depends only on the set of eigenvalues and resonances, so we have a single function \(P(s)\) for all \(g_k\). Furthermore, Proposition 5.8 tells us that the the polynomial \(q(s)\) in the relation,

\[
D_{g_k,\tau_k}(s) = e^{q(s)} \frac{P(s)}{Z_{\tau_k}(s)Z_\infty(s)},
\]

(6.1)
is also independent of \( k \). (Recall that the factor \( Z_\infty(s) \) depends only on \( \chi(X) \) and so has no \( k \) dependence either.) In addition to (6.1), we have also the invariance of the relative heat invariants \( a_j = a_j(\varphi_k, \tau_k) \). And lastly, we have the Polyakov formula expressing \( \log D_{g_\tau, \tau_k}(1) \) in terms of \( \varphi_k \) and \( \tau_k \) (but note that \( \log D_{g_\tau, \tau_k}(1) \) is not independent of \( k \)).

**Proof of Theorem 1.4.** By combining the Polyakov formula (Proposition 1.2) with (6.1) we obtain

\[
-\frac{1}{6\pi} \int_X \left( \frac{1}{2} |\nabla \varphi_k|^2 - \varphi_k \right) \, d\tau_k = c - \log Z_{\tau_k}(1),
\]

where \( c = q(1) + \log P(1) - \log Z_\infty(1) \) which is independent of \( k \). Solving for \( \varphi_k \) gives

\[
\int_X \varphi_k \, d\tau_k = \frac{1}{2} \int_X |\nabla \varphi_k|^2 \, d\tau_k + 6\pi[c - \log Z_{\tau_k}(1)].
\]

Inspection of the definition (3.1) of \( Z_\tau \) yields that \( -\log Z_{\tau_k}(1) \geq 0 \), and hence

\[
\int_X \varphi_k \, d\tau_k \geq 6\pi c.
\]

On the other hand, recall that the zeroth heat invariant,

\[
a_0 = \frac{1}{4\pi} \int_X (e^{2\varphi_k} - 1) \, d\tau_k,
\]

is also a constant. An application of Jensen’s inequality then gives

\[
\int_X \varphi_k \, d\tau_k \leq \frac{1}{2} \log(4\pi a_0 + 1).
\]

The left-hand side of (6.2) is thus bounded above and the right-hand side is bounded below. The conclusion is a set of bounds:

\[
6\pi c \leq \int_X \varphi_k \, d\tau_k \leq \frac{1}{2} \log(4\pi a_0 + 1),
\]

(6.3)

\[
\int_X |\nabla \tau_k \varphi_k|^2 \, d\tau_k \leq \log(4\pi a_0 + 1) - 12\pi c,
\]

(6.4)

and

\[
0 \leq -\log Z_{\tau_k}(1) \leq \frac{1}{12\pi} \log(4\pi a_0 + 1) - c.
\]

(6.5)

By Corollary A.2, the bound (6.3) implies that a subsequence of the \( \tau_k \)'s converges, up to pull-back by diffeomorphisms, in the \( C^\infty \) topology on \( X \) to some hyperbolic metric \( \tau_\infty \). Note that, by the construction made in the proof of Theorem A.1, we see that each of these diffeomorphisms preserves the respective pants decompositions. Therefore the assumption \( K = \tilde{X}_{\tau_k} \) is preserved through the pullbacks. Assume that a convergent subsequence has been chosen and diffeomorphisms applied to give convergence in \( C^\infty(X) \). For simplicity we will continue to denote the sequence by \( \tau_k \).

On the ends \( F_j = (0, \infty)_t \times S^1_\theta \) these metrics can all be written as

\[
d\tau_k|_{F_j} = dt^2 + \ell_{k,j}^2 \cosh^2 t \, d\theta^2.
\]

The topology of \( \rho^{-2}C^\infty(\tilde{X}) \) is independent of the choice of \( \rho \). With the choice \( \rho = e^{-t} \), convergence of \( \tau_k \) in \( \rho^{-2}C^\infty(\tilde{X}) \) follows from the convergence \( \ell_{k,j} \to \ell_{\infty,j} \) for all \( j \).
It remains to show that \{\varphi_k\} has a convergent subsequence. In the estimates above, the \(C^\infty\) convergence of \(\tau_k\) allows us to replace \(\tau_k\) by \(\tau_\infty\) and still obtain uniform bounds. From (3.3) and (3.4) we derive estimates,

\[
\left| \int K \varphi_k \ d\tau \right| \leq C, \quad \int K |\nabla_{\tau_\infty} \varphi_k|^2 \ d\tau \leq C,
\]

which allow us to bound \(\varphi_k\) uniformly in \(H^1(X)\). From this point on the analysis is exactly as in \S 2 of Osgood-Phillips-Sarnak [32]. The constancy of the second relative heat invariant \(\langle 4.4 \rangle\), in conjunction with the estimates above, is used to bound \(\varphi_k\) uniformly in \(C^0(X)\). Then the higher heat invariants \(\langle 4.7 \rangle\) allow a bootstrap argument extending the uniform bounds to \(H^m(X)\) for all \(m\). This, together with the restriction \(\varphi_k \in C_0^\infty(K)\), implies compactness of the \(\varphi_k\)'s in the \(C_0^\infty(K)\) topology, and completes the proof.

**Appendix A. Finiteness and properness for hyperbolic surfaces**

Let \(S\) be an oriented compact topological surface \(S\) with boundary \(\partial S\). Let \(\mathcal{M}(S)\) denote the set of equivalence classes (up to isometry) hyperbolic metrics \(h\) on \(S\) such that \(\partial S\) is geodesic with respect to \(h\).

We will use the \(C^\infty\) topology on \(\mathcal{M}(S)\): A sequence \(\{h_n\} \subset \mathcal{M}(S)\) converges if and only if there exists \(h'_n \in h_n\) such that the coordinate components of \(h'_n\) converge in \(C^\infty\).

The (primitive) length spectrum \(\Lambda(h)\) does not depend on the choice of representative for the class \(h\). Hence the class \(h\) defines through (3.1) a dynamical zeta function \(Z_h(s)\). Note also that the total length, \(\ell_h(\partial S)\), of the boundary does not depend on the representative of \(h\).

**Theorem A.1.** Let \(R > 0\). The set of all \(h \in \mathcal{M}(S)\) such that \(-\log Z_h(1) \leq R\) and \(\ell_h(\partial S) \leq R\) is compact.

The geometry of a convex co-compact hyperbolic surface \((X, \tau)\) is determined by the geometry of its convex core. Moreover, the zeta function associated to \((X, \tau)\) equals the zeta function associated to its convex core. Therefore, we have the following:

**Corollary A.2.** Let \(X\) be an open surface of finite topological type, with \(\tau_n\) a sequence of complete hyperbolic metrics on \(X\). If \(-\log(Z_{\tau_n}(1))\) is bounded from above, then there exists a hyperbolic metric \(\tau_\infty\) and a sequence of diffeomorphisms \(\phi_n : X \to X\) such that \(\phi_n^{*}(\tau_n)\) converges to \(\tau_\infty\) in the \(C^\infty\) topology.

Before considering the proof of Theorem A.1, we give provide an example that shows that an upper bound on \(\ell(\partial S)\) is a necessary condition.

**Example A.3.** Let \(S\) be the sphere with three open discs removed, a topological pair of pants. Then each \(h \in \mathcal{M}(S)\) is determined by the lengths \(\ell(\gamma_1), \ell(\gamma_2), \ell(\gamma_3)\), of the connected components \(\gamma_1, \gamma_2, \gamma_3\) of \(\partial S\). Conversely, given \((\ell_1, \ell_2, \ell_3) \in (\mathbb{R}^+)^3\) there exists \(h(\ell_1, \ell_2, \ell_3)\) such that \(\ell(\gamma_i) = \ell_i\). (See, for example, Theorem 3.1.7 [1].)

We claim that

\[
\inf \Lambda(h(\ell_1, \ell_2, \ell_3)) = \inf \{\ell_1, \ell_2, \ell_3\}
\]

and hence if \(\ell_1, \ell_2, \ell_3\) are bounded from below then \(-\log(Z_{h(\ell_1, \ell_2, \ell_3)})\) is bounded from above. On the other hand, the family \(h(\ell, \ell, \ell)\), for example, has no limit as
∂S converges in a subsequence, still called α.

Theorem A.1.

The idea of the proof comes from Theorem 5.2.3 of [11]. Let \( \exp : [0, \infty) \times \partial S \rightarrow S \) be the exponential map associated to the normal bundle of \( \partial S \). Note that on each connected component of \( K_i \), there exists a constant \( c = c(S) \) such that the restriction of \( \exp \) to \([0, t) \times \partial S\) is injective. Note that on each connected component of \( K_i \), \( [0, \infty) \times \partial S \) we have \( \exp^*(h) = d\rho^2 + \ell_i^2 \cosh(\rho)^2 d\theta_i^2 \) where \( \ell_i \) is the length of \( \partial K_i \). Hence,

\[
\begin{align*}
\ell(\exp([0, t) \times \partial S)) &= \cosh(t) \cdot \ell(\partial S) \\
Area(\exp([0, t) \times \partial S)) &= \sinh(t) \cdot \ell(\partial S).
\end{align*}
\]

From \( \sinh^2(t) + 1 = \cosh^2(t) \), we then find that

\[
\ell^2(\exp([0, t) \times \partial S)) \leq \ell^2(\partial S) + (\text{Area}(S))^2.
\]

The set \( \exp([0, t) \times \partial S) \) is a union of simple closed curves each of which is freely homotopic to a unique simple closed geodesic \( \gamma_i \). Let \( P_1, \ldots, P_k \) be the connected components of \( S \setminus (\cup \gamma_i) \) that are pairs of pants. By construction, \( k \geq 1 \) unless \( S \) is a pair of pants—see p. 126-129 [11]—and \( \ell(S \setminus (\cup P_i)) \leq \ell(\cup \gamma_i) \leq \ell(\exp([0, t) \times \partial S)) \). Therefore, by (A.3)

\[
\ell^2(\partial S \setminus (\cup P_i)) \leq \ell^2(\partial S) + (\text{Area}(S))^2.
\]

Since \( k \geq 1 \), we have \( \text{Area}(S \setminus (\cup P_i)) \leq \text{Area}(S) - 2\pi \) and since the claim is vacuous for a pair of pants, the general claim follows by induction.

Proof of Theorem A.4. Suppose that \( - \log(Z_h(1)) \leq R \). Since each term of (A.1) is positive and decreasing in \( s > 0 \), we find that \( - \log(1 - \exp(-(2 + k) \cdot \ell(\gamma))) < R \) for all \( \gamma \) and \( k \). In particular, for each primitive—and hence every—closed geodesic \( \gamma \)

\[
\ell(\gamma) \geq \varepsilon_R > 0
\]

where \( \varepsilon_R = -\frac{1}{2} \ln(1 - \exp(-R)) \).

Let \( h_n \) be a sequence of hyperbolic metrics on \( S \) inducing geodesic boundary and satisfying \( \inf \Lambda(h_n) \geq \varepsilon_R \) and \( \ell(\partial S) < R \). It suffices to show that there exists a subsequence, still called \( h_n \), and diffeomorphisms \( \psi_n : S \rightarrow S \) such that \( \psi_n^*(h_n) \) converges in \( C^\infty \).

Applying Theorem A.3 to each \( h_n \) gives an infinite sequence of pants decompositions \( \{P_1^n, \ldots, P_k^n\} \) with \( \ell_{h_n}(\partial P_i) < c \) for all \( n \) and \( i \). Since there are only finitely
many combinatorial types of pants decompositions—see [11] §3.6—we may assume without loss of generality that \( \{P_1^n, \ldots, P_k^n\} \) has constant combinatorial type. It follows that there exist diffeomorphisms \( \phi_n : S \to S \) such that \( \phi_n(P_i^n) = P_i^1 \) for \( i = 1, \ldots, k \). Hence, by pulling back \( h_n \) by \( \phi_n \), we may assume without loss of generality that each \( h_n \) gives the same pants decomposition \( \{P_1, \ldots, P_n\} \).

By Theorem A.4 and (A.3), we have \( \varepsilon_R \leq \ell_{h_n}(\gamma_{ij}) \leq c \) for each boundary component \( \gamma_{ij} \) of \( \partial P_i, j = 1, 2, 3 \). Therefore there is a subsequence, still denoted \( h_n \), such that each \( \ell_{h_n}(\gamma_{ij}) \) converges as \( n \) tends to infinity. Thus, by the discussion in Example A.3 and the reference given there, there exist diffeomorphisms \( \psi_n : P_i \to P_i^1 \) such that \( \psi_n^*(h_n|_{P_i}) \) converges to a metric on \( P_i \) in \( C^\infty \).

Finally, by perturbing each \( \psi_n \) in a collar neighborhood of each boundary component of \( P_i \), one can construct a suitable diffeomorphism of the entire surface \( \psi_n : S \to S \). Moreover, the associated twist angles (see [11] §3.3.) are bounded, and hence by passing to a further subsequence if necessary, we may assume that \( \psi_n^*(h_n) - (\psi_n^*)^*(h_n) \) is Cauchy. It follows that \( \psi_n^*(h_n) \) converges in \( C^\infty \) to a metric \( h \) on \( S \).

Our variant of Bers’ theorem also has the following corollary. Compare with Theorem 13.1.3 in [11].

**Theorem A.5.** Let \( h_0 \in \mathcal{M}(s) \) and \( R > 0 \). Then the set of all \( h \in \mathcal{M}(S) \) such that \( \Lambda(h) = \Lambda(h_0) \) and \( \ell_{h}(\partial S) \leq R \) is finite.

**Proof.** The set in question is compact by Theorem A.4. The twist angles are determined by \( \Lambda(h_0) \) as are the lengths of the boundary components of the \( P_i \). Hence the set is discrete as well as compact.

**Appendix B. Resolvent Construction and Estimates**

We briefly review the construction of the resolvent carried out in Guillopé-Zworski [17] since we will used some detailed information from that construction to prove key estimates on the relative determinant. We follow closely the outline of [17]. The results of this section apply to any metric that is hyperbolic near infinity, but not to a general asymptotically hyperbolic metric. For notational simplicity, let us assume \( \tau \) is a hyperbolic metric on \( X \) and \( g \) is a perturbation that is equal to \( \tau \) on \( \hat{X} \).

The cylinders \( F_j \) with the hyperbolic metric \( \tau \) are isometric to hyperbolic half-cylinders \( F_j^0 = (0, \infty) \times S^1_g \) with metric

(B.1) \[ \tau_j^0 = dt^2 + \ell_j^2 \cosh^2 t \, d\theta^2 \]

where \( \ell_j \) is the geodesic length of the circle at \( t = 0 \) (which is a closed geodesic). If \( \Delta_{F_j^0} \) is the Laplacian on \( (F_j^0, \tau_j^0) \), we let

\[ R_{F_j^0}(s) = (\Delta_{F_j^0} - s(1 - s))^{-1} \]

be the resolvent for the hyperbolic Laplacian on \( F_j^0 \) with Dirichlet boundary conditions at \( t = 0 \). This resolvent can be computed explicitly (see for example [14] or [13]) and is known to have poles contained in the set of \( \zeta_{n,k} = -k + 2\pi in/\ell_j \) where \( n \) is any integer and \( k = 0, 1, 2, \cdots \). In particular, \( R_{F_j^0}(s) \) is entire in any half-plane \( \text{Re}(s) > \varepsilon \). Finally, it follows from explicit formulas that if \( \chi \) and \( \psi \) are smooth, compactly supported functions in \( F_j^0 \) with disjoint supports, \( \chi R_{F_j^0}(s)\psi \)
has a smooth kernel with derivatives bounded uniformly in $s$ with $\text{Re}(s) > \varepsilon$ for any fixed $\varepsilon > 0$.

First we describe the parametrix constructed from model operators. Let $\eta \in C^\infty(\mathbb{R})$ with $\eta(t) = 1$ for $t < 1/3, \eta(t) = 0$ for $t > 2/3$, and let $\eta_a(t) = \eta(t - a)$. We will pick $a > 1$ in what follows. Let $\chi_a \in C^\infty_c(X)$ with $\chi_a = 1$ on $Z$ and $\chi_a = \eta_a$ on each funnel $F_j$ (referring to the coordinates $(t, \theta)$ as in (B.1)). We denote by $\chi_{a,j}$ the restriction of $\chi_a$ to $F_j$. For a fixed $a > 0$ and real $s_0$ with $s_0 > 1$, we set
\[(B.2)\]
$$E_g(s) = Q_0(s_0) + Q(s)$$
where
\[(B.3)\]
$$Q_0(s_0) = \chi_{a+2} R_g(s_0) \chi_{a+1}$$
and
\[(B.4)\]
$$Q(s) = \sum_{j=1}^M (1 - \chi_{a,j}) J_j^c R_{\pm_0}^c(s) J_j (1 - \chi_{a+1,j}).$$
Here $J_j : F_j \to F_j^0$ is an isometry mapping the funnel end to the model manifold.

It is not difficult to compute that
$$\Delta_g - s(1 - s)) E_g(s) = I + L(s_0, s)$$
where
\[(B.5)\]
$$L(s_0, s) = (s_0(1 - s_0) - s(1 - s)) Q_0(s_0) + [\Delta_g, \chi_{a+2}] R_g(s_0) \chi_a - \sum_{j=1}^M [\Delta_g, \chi_a] J_j^c R_{\pm_0}^c(s) J_j (1 - \chi_{a+1,j}).$$
The first right-hand term is a Hilbert-Schmidt integral operator with compactly supported kernel, while the second is a compactly supported operator with smooth kernel owing to the fact that the derivatives of $\chi_{a+2}$ and $\chi_a$ have disjoint supports. The third term in $L(s_0, s)$ is a sum of operators with kernels belonging to $(\rho')^* C^\infty(\bar{X} \times \bar{X})$ having compact support in the first variable (here $\rho'$ is a defining function for $\partial \bar{X}$ in the second variable). To construct the resolvent $R_g(s)$ in the half plane $\text{Re}(s) > 1/2 - N$, one inverts the operator $(I + L(s_0, s))$ viewed as a map from $\rho^N L^2(X)$ to itself, using the analytic Fredholm theorem.

One then obtains
\[(B.6)\]
$$R_g(s) = E_g(s)(I + L(s_0, s))^{-1} = E_g(s) - E_g(s)(I + L(s_0, s))^{-1} L(s_0, s).$$

**Proof of Lemma 5.4.** It suffices to show that a similar estimate holds for $H(s)$ as defined in (3.3) since the desired estimate follows by integration. From (3.6) we may write
$$R_g(s) - R_+(s) = E_g(s) - E_+(s)$$
\begin{align*}
&+ (E_g(s) - E_+(s))(I + L(s_0, s))^{-1} L(s_0, s) \\
&+ E_+(s)(I + L(s_0, s))^{-1} (L(s_0, s) - L_0(s_0, s)) \\
&+ E_+(s) \left[ (I + L(s_0, s))^{-1} - (I + L_0(s_0, s))^{-1} \right] L_0(s_0, s) \\
= T_1(s) + T_2(s) + T_3(s) + T_4(s).
\end{align*}
It suffices to estimate the zero-trace of the quantities $T_i(s) - T_i(s_0)$. As we will see,
the difference is important only when \( i = 1, 2 \) and otherwise it is possible to prove polynomial bounds on the zero-traces of \( T_i(s), i = 3, 4 \). Indeed, \( T_1(s) - T_1(s_0) \), \( T_2(s) - T_2(s_0) \), and \( T_3(s) \) are operators with continuous kernel whose support is compact in at least one of the variables; thus the zero-trace is actually a trace:

\[
0\text{-tr}_T [T_i(s) - T_i(s_0)] = \text{tr} [\chi_{a+3} [T_i(s) - T_i(s_0)] \chi_{a+3}], \ i = 1, 2
\]

\[
0\text{-tr}_T [T_3(s)] = \text{tr} [\chi_{a+3} T_3(s) \chi_{a+3}].
\]

Thus it suffices to estimate the trace norms of the operators

\[
\chi_{a+3} [T_1(s) - T_1(s_0)] \chi_{a+3}, \ \chi_{a+3} [T_2(s) - T_2(s_0)] \chi_{a+3}, \ \chi_{a+3} T_3(s) \chi_{a+3}.
\]

We will use a separate argument to estimate \( T_4 \). In what follows, \( \| \cdot \|, \| \cdot \|_1, \) and \( \| \cdot \|_2 \) will denote respectively the operator norm, trace norm, and Hilbert-Schmidt norm.

The \( T_1(s) \) term is, by \( \text{(B.2)}, \text{(B.3)}, \) and \( \text{(B.4)} \), equal to

\[
\chi_{a+2} [R_g(s_0) - R_{\tau}(s_0)] \chi_{a+1}
\]

since the model operators occurring in \( Q(s) \) are the same for the two problems. It follows that \( T_1(s) - T_1(s_0) = 0 \) so this term makes no contribution.

To estimate the remaining terms, we first note that

\[
(I + L(s_0, s))^{-1}L(s_0, s)\chi_{a+3} = (I + K(s_0, s))^{-1}K(s_0, s)
\]

where

\[
K(s_0, s) = L(s_0, s)\chi_{a+3}
\]

and similarly for \( L \) and \( K \) replaced by \( L_0 \) and \( K_{\tau} \). It is not difficult to see that

\[
\text{(B.7)} \quad \|K(s_0, s)\|_2 \leq C_\varepsilon(s^2)
\]

for \( s \in S_\varepsilon \), with a similar estimate for \( K_{\tau} \). From \cite{17}, Lemma 3.6, we have the estimate

\[
\text{(B.8)} \quad \| (I + K(s_0, s))^{-1} \| \leq \exp(C_\eta(s^{2+\eta}))
\]

for any \( \eta > 0 \) and all \( s \in S_\varepsilon \setminus \bigcup_j D_j \) (The statement of \cite{17}, Lemma 3.6 also excludes singularities of the model resolvents on the half-cylinders \( F^j \), but these singularities lie in the half-plane \( \text{Re}(s) \leq 0 \) and so are already excluded from \( S_\varepsilon \)).

We now estimate the remaining terms. First of all,

\[
\chi_{a+3} T_2(s) \chi_{a+3} = \chi_{a+2} [R_g(s_0) - R_{\tau}(s_0)] \chi_{a+1} \times (I + K(s_0, s))^{-1}K(s_0, s)
\]

but the difficulty here is that the difference

\[
R_g(s_0) - R_{\tau}(s_0)
\]

is not trace-class. On the other hand, the difference

\[
\chi_{a+3} [T_2(s) - T_2(s_0)] \chi_{a+3} = \chi_{a+2} [R_g(s_0) - R_{\tau}(s_0)] \chi_{a+1} \times (I + K(s_0, s))^{-1}K(s_0, s) - K(s_0, s_0) \times (I + K(s_0, s))^{-1}
\]

and the factor \( K(s_0, s) - K(s_0, s_0) \) is easily seen to be a trace-class operator whose trace norm has at most polynomial growth in \( s \). It now follows from \( \text{(B.7)}, \text{(B.8)} \), and the fact that the first factor is a fixed trace-class operator that

\[
\| \chi_{a+3} [T_2(s) - T_2(s_0)] \chi_{a+3} \| \leq \exp(C'\eta(s^{2+\eta})).
\]
To bound $T_3$, we note that

$$\chi_{a+3}T_3(s)\chi_{a+3} = (\chi_{a+3}E_\tau(s)\chi_{a+4})(I + K(s_0, s))^{-1} [K(s_0, s) - K_\tau(s_0, s)]$$

and use the fact that $\varepsilon$ of the terms in the last two lines of (B.10) is actually trace-class since $L\psi L\chi W$ we have inserted the cutoff functions $\chi$ similarly characteristic function of the set $\rho \chi$.

We can now use (B.7) and (B.8) together with (B.9) to conclude that

$$\|\chi_{a+3}T_3(s)\chi_{a+3}\|_1 \leq \exp(C'_\eta(s)^{2+\eta}).$$

To bound $T_4(s)$ we make a slightly different argument since the integral kernel need not have compact support. Instead, we consider $\chi\varepsilon T_4(s)\chi\varepsilon$ where $\chi\varepsilon$ is the characteristic function of the set $\rho \geq \varepsilon$. Write $E_\tau$ for $E_\tau(s)$, $L$ for $L(s_0, s)$ and similarly $L_0$ for $L_0(s_0, s)$. We compute, by cyclicity of the trace,

$$\text{tr}(\chi\varepsilon T_4(s)\chi\varepsilon) = \text{tr}(\chi\varepsilon E_\tau [(I + L)^{-1} - (I + L_0)^{-1}] L_0\chi\varepsilon)$$

$$= \text{tr}((I + L_0)^{-1}L_0\chi^2 E_\tau (I + L)^{-1}(L_0 - L))$$

$$= \text{tr}(\chi_{a+5/2}L_0(I + L_0)^{-1}L_0\chi^2 E_\tau (I + L)^{-1}L_0\chi_{a+3})$$

$$= \text{tr}(\chi_{a+5/2}(I + L_0)^{-1}L_0\chi^2 E_\tau (I + L)^{-1}L_0\chi_{a+3})$$

We have inserted the cutoff functions $\chi_{a+5/2}$ on the left owing to the mapping properties of $L$ and $L_0$ and on the right by cyclicity of the trace. Observe that each of the terms in the last two lines of (B.10) is actually trace-class since $L\psi$ and $L_0\psi$ are Hilbert-Schmidt for any function $\psi \in C_0^\infty(X)$. Moreover, $L_0\chi^2 E_\tau = L_0\chi_{a+3}E_\tau$ for sufficiently small $\varepsilon$ since $\chi_{a+3} E_\tau = E_\tau$, so that

$$0\text{-tr}_\tau(T_4(s)) = \text{tr}(\chi_{a+5/2}L_0(I + L_0)^{-1}L_0\chi_{a+3}E_\tau (I + L)^{-1}L_0\chi_{a+3})$$

$$+ \text{tr}(\chi_{a+5/2}(I + L_0)^{-1}L_0\chi_{a+3}E_\tau (I + L)^{-1}L_0\chi_{a+3})$$

$$= \text{tr}(\chi_{a+3}L_0(I + K_\tau)^{-1}K_\tau\chi_{a+4}E_\tau (I + K)^{-1}K_\tau)$$

$$+ \text{tr}(\chi_{a+3}(I + K_\tau)^{-1}K_\tau\chi_{a+4}E_\tau (I + K)^{-1}K_\tau).$$

We can now use (B.7) and (B.8) together with (B.9) to conclude that

$$|0\text{-tr}_\tau(T_4(s))| \leq \exp(C'_\eta(s)^{2+\eta})$$

and complete the proof. \qed

**Appendix C. Logarithmic derivative of the relative scattering operator**

To prove the formula (C.6) we need to explicitly evaluate the 0-traces appearing in $F(s)$. We will employ the basic strategy used in §6.1 of [3]; similar calculations are done in Proposition 4.1 of [7]. For the hyperbolic metric $\gamma$ we have an identity:

$$R_\gamma(s; z, z') - R_\gamma(1 - s; z, z') = -\frac{1}{2s - 1} \int_{\partial X} E_\gamma(1 - s; z, y) E_\gamma(s; z', y) dh(y),$$

(C.1)
where $dh$ is the Riemannian density induced on $\partial X$ by $\rho^2 \tau |_{\partial X}$. Note that the same density is induced by $\rho^2 g |_{\partial X}$. Here $E_r(s; z, y)$ is the Poisson kernel for $\Delta_r + s(s - 1)$, which can be realized as a limit

$$E_r(s; z, y) = (2s - 1)2^{2s-1} \frac{\Gamma(s - 1/2)}{\Gamma(1/2 - s)} \rho(z')^{-s} R_r(s; z, z')|_{z' = y}.$$  

If $R_g(s; z, z)$ is the kernel of the resolvent of $\Delta_g$, with respect to the measure $d\tau$, and $\hat{E}_g(s; z, y)$ is defined by a limit as above, then it is simple to check that the corresponding relation holds.

With these relations we can write

$$F(s) = \text{FP}_{s \rightarrow 1} \int_{\rho \geq \varepsilon} \int_{\partial X} \left[ \hat{E}_g(1 - s; z, y) \hat{E}_g(s; z, y) 
- E_r(1 - s; z, y) E_r(s; z, y) \right] dh(y) d\tau(z).$$  

(C.2)

Technically we should assume here that $\rho$ is a suitable defining function for the definition of $0-tr_r$. However, the calculation will show that $F(s)$ does not actually depend on the choice of defining function.

The next step is to transform the integral using the Maass-Selberg relation, which for $E_r(s)$ reads:

$$(2s - 1) \int_{\rho \geq \varepsilon} \int_{\partial X} E_r(1 - s; z, y) E_r(s; z, y) dh(y) d\tau(z)$$
$$= - \int_{\rho = \varepsilon} \int_{\partial X} \left[ E_r(1 - s; z, y) \partial_s \partial_\nu E_r(s; z, y) 
- \partial_\nu E_r(1 - s; z, y) \partial_s E_r(s; z, y) \right] dh(y) d\sigma_{\varepsilon}(z),$$

where $d\sigma_{\varepsilon}$ is the measure induced on $\{ \rho = \varepsilon \}$ by $\tau$, and $\partial_\nu$ is the inward Riemannian normal to $\{ \rho = \varepsilon \}$.

Since the corresponding equation for $\hat{E}_g$ contains extra factors, we will go through the proof for this case. Let $\omega(s, t) = (s + t)(1 - s - t) - s(1 - s)$. The first step is to use the property $(\hat{E}_g - s(1 - s)) \hat{E}_g(s; z, y) = 0$ to write

$$(2s - 1) \int_{\rho \geq \varepsilon} \int_{\partial X} \hat{E}_g(1 - s; z, y) \hat{E}_g(s; z, y) dh(y) d\tau(z)$$
$$= \lim_{t \rightarrow 0^+} \frac{2s - 1}{\omega(s, t)} \int_{\rho \geq \varepsilon} \int_{\partial X} \left[ \hat{E}_g(1 - s; z, y) \hat{E}_g(s + t; z, y) 
- \hat{E}_g(1 - s; z, y) \hat{E}_g(s + t; z, y) \right] dh(y) d\tau(z).$$

Then integrate by parts, using $\hat{E}_g = e^{-\varphi} \Delta_r e^{-\varphi}$:

$$= \lim_{t \rightarrow 0^+} \frac{2s - 1}{\omega(s, t)} \int_{\rho \geq \varepsilon} \int_{\partial X} \left[ e^{-\varphi(z)} \hat{E}_g(1 - s; z, y) \partial_\nu (e^{-\varphi(z)} \hat{E}_g(s + t; z, y)) 
- \partial_\nu (e^{-\varphi(z)} \hat{E}_g(1 - s; z, y)) e^{-\varphi(z)} \hat{E}_g(s + t; z, y) \right] dh(y) d\sigma_{\varepsilon}(z).$$

$$= - \int_{\rho = \varepsilon} \int_{\partial X} \left[ e^{-\varphi(z)} \hat{E}_g(1 - s; z, y) \partial_\nu (e^{-\varphi(z)} \partial_s \hat{E}_g(s; z, y)) 
- \partial_\nu (e^{-\varphi(z)} \hat{E}_g(1 - s; z, y)) e^{-\varphi(z)} \partial_s \hat{E}_g(s; z, y) \right] dh(y) d\sigma_{\varepsilon}(z).$$

Since we know that $(\partial_\nu - s) E_r(s; z, y)$ is of lower order as $z \rightarrow \partial X$, it is useful to extract terms of this form as we substitute the Maass-Selberg relations back into
For this purpose we note that $\partial_s \partial_v = \partial_s (\partial_v s) + s \partial_s + 1$. We will break the resulting formula for $F(s)$ up into pieces:

$$F(s) = \frac{1}{2s - 1} \int_{\partial X} \left( \text{FP}_{\xi \rightarrow 0} \int_{\rho = \xi} [J_1(z, y) + J_2(z, y) + J_3(z, y)] d\sigma(z) \right) dh(y),$$

where

$$J_1(s; z, y) = e^{-2\varphi(z)} \tilde{E}_g(1 - s; z, y) \tilde{E}_g(s; z, y) - E_\tau(1 - s; z, y) E_\tau(s; z, y)$$

$$J_2(s; z, y) = e^{-\varphi(z)} \tilde{E}_g(1 - s; z, y) \partial_s (\partial_s (1 - s; z, y) e^{-\varphi(z)} \partial_s \tilde{E}_g(s; z, y)$$

$$- (\partial_s - 1) e^{-\varphi(z)} \tilde{E}_g(1 - s; z, y) e^{-\varphi(z)} \partial_s \tilde{E}_g(s; z, y)$$

$$- E_\tau(1 - s; z, y) \partial_s E_\tau(s; z, y)$$

$$J_3(s; z, y) = 2e^{-2\varphi(z)} \tilde{E}_g(1 - s; z, y) \tilde{E}_g(1 - s; z, y)$$

$$- (2s - 1) e^{-\varphi(z)} \tilde{E}_g(1 - s; z, y) \partial_s E_\tau(s; z, y)$$

Let $I_j(s)$ denote the contribution to $F(s)$ from $J_j$. We will show that $I_1(s) = I_2(s) = 0$, while $I_3(s)$ is equal to the stated result.

To handle $J_1$ we reverse the identity (C.1) and write

$$- \frac{1}{2s - 1} \int_{\partial X} J_1(z, y) dh(y) = - e^{-2\varphi(z)} \left[ \tilde{R}_g(s; z, z') - \tilde{R}_g(1 - s; z, z') \right]_{z' = z}$$

$$- R_\tau(s; z, z') - R_\tau(1 - s; z, z') \right]_{z' = z},$$

where we note that $R_\tau(s) - R_\tau(1 - s)$ has a continuous kernel and so can be evaluated on the diagonal. In computing the finite part for $I_1(s)$, the factor $e^{-2\varphi(z)}$ can be replaced by 1, since $1 - e^{-2\varphi(z)} = O(\rho^2)$. So at this stage we have

$$I_1(s) = - \text{FP}_{\xi \rightarrow 0} \int_{\rho = \xi} [\tilde{R}_g(s; z, z') - \tilde{R}_g(1 - s; z, z')$$

$$- R_\tau(s; z, z') + R_\tau(1 - s; z, z') \right]_{z' = z} d\sigma(z).$$

Analysis as in Theorem 3.1 of [17] shows the integrand can by split into two components lying in $\rho^2C^0(X)$ and $\rho^sC^\infty(X)$. The first contributes zero as $\varepsilon \rightarrow 0$, while the second contributes zero for $\text{Re}(s) > 1$. Thus $I_1(s)$ vanishes for all $s$ by meromorphic continuation.

The analysis of $I_2(s)$ is based entirely on the fact that $(\partial_s - 1) E_\tau(s; z, y)$ vanishes to higher order as $z \rightarrow \partial X$ than $E_\tau(s; z, y)$. So the conclusion that $I_2(s) = 0$ follows exactly as in Lemma 6.5 of [33].

The final step is to evaluate

$$I_3(s) = - \int_{\partial X} \text{FP}_{\xi \rightarrow 0} \int_{\rho = \xi} \left[ e^{-2\varphi(z)} \tilde{E}_g(1 - s; z, y) \partial_s E_\tau(s; z, y)$$

$$- E_\tau(1 - s; z, y) \partial_s E_\tau(s; z, y) \right] d\sigma(z) dh(y).$$

We will need the relation between Poisson and scattering kernels:

$$S_\tau(s; y, y') = [\rho(z)^{-s} E_\tau(s; y, y')]_{z = y},$$

for $y \neq y'$. For convenience let $\psi_\tau(s; z, y) = \rho(z)^{-s} E_\tau(s; z, y')$. In the formula for $I_3$ above, the $e^{-2\varphi(z)}$ may be replaced by 1 as in the analysis of $I_1$. Further more, after substituting $E_\tau = \rho^s \phi_\tau$, the terms involving $\partial_s \rho^s$ are logarithmic and thus do
not contribute to the finite part. This substitution thus gives
\[
I_3(s) = -\int_{\partial X} \mathcal{F}p_{\epsilon 10} \int_{\rho=\epsilon} \left[ \psi(1-s; z, y) \partial_s \psi(s; z, y) \right. \\
- \left. \psi_0(1-s; z, y) \partial_s \psi_0(s; z, y) \right] \rho \sigma_\epsilon(z) dh(y).
\]
(C.3)
As pointed in §5.1, \( S_g(s) - S_\tau(s) = T(s) \) is of order \( 2 \Re(s) - 3 \). We can write
\[
S_g(1-s)S'_g(s) - S_\tau(1-s)S'_\tau(s) = T(1-s)S'_g(s) + S_\tau(1-s)T'(s),
\]
so this combination is of order \(-2\) and has a continuous kernel. To take the limit \( \epsilon \to 0 \) in \( I_3 \), we note that \( \rho \sigma_\epsilon \to dh \) as \( \rho \to 0 \), and the integrand in brackets in (C.3) approaches the kernel (C.4). Thus the limit is
\[
F(s) = I_3(s) = -\tr[S_g(1-s)S'_g(s) - S_\tau(1-s)S'_\tau(s)] = -\frac{d}{ds} \log \det S_{g, \tau}(s).
\]

Appendix D. Isopolar Surfaces, by Robert Brooks

In this appendix, we briefly review the construction of isopolar manifolds via the Sunada method (see [6] and [8]), and then present a survey of various kinds of isopolar surfaces that can be constructed in this way. In fact, the examples are isophasal, meaning they have identical scattering phase. Isophasal implies isopolar, but the eigenvalues and resonances determine the scattering phase only up to finitely many parameters.

Let \( G \) be a finite group, and let \( H_1 \) and \( H_2 \) be subgroups of \( G \) satisfying the following condition, known as the Sunada condition:
\[
\text{for all } g \in G, \#([g] \cap H_1) = \#([g] \cap H_2),
\]
(D.1)
where \([g]\) denotes the conjugacy class of \( g \) in \( G \).

Let \( L^2(G) \) denote the vector space of functions on \( G \) with the left action of \( G \) given by
\[
(g^*(f))(x) = f(g^{-1}x)
\]
and right action given by
\[
f^g(x) = f(xg).
\]

It is shown, for instance in [6] or [8], that (D.1) is equivalent to
\[
(L^2(G))^{H_1} \text{ is } G\text{-equivalent to } (L^2(G))^{H_2},
\]
(D.2)
where \((L^2(G))^{H_i}\) denotes the subspace of \( L^2(G) \) which is invariant under the left action of \( H_i \).

Condition (D.2) is by definition equivalent to the existence of a function
\[
c : G \to \mathbb{R}
\]
such that the \( G \)-equivariant map
\[
T : L^2(G) \to L^2(G)
\]
given by
\[
T(f)(x) = \sum_{g \in G} c(g) f(gx)
\]
satisfies
\[
T \text{ is an isomorphism from } (L^2(G))^{H_1} \text{ to } (L^2(G))^{H_2}.
\]
(D.3)
Note that the condition that the image of $T$ lies in $(L^2(G))^{H_2}$ is precisely that the function $c$ satisfies
\[ c(gh) = c(g) \text{ for } h \in H_2. \]

We then have:

**Theorem D.1.** Let $S_0$ be a surface with funnels or cusps, and let $(G, H_1, H_2)$ satisfy (D.4) (or, equivalently, D.5).

Let $\phi: \pi_1(S_0) \to G$

be a homomorphism, and let $S_1$ and $S_2$ be the covers of $S_0$ whose fundamental groups are $\phi^{-1}(H_1)$ and $\phi^{-1}(H_2)$ respectively.

Then $S_1$ and $S_2$ are isophasal.

**Sketch of proof:** (see [6] and [8] for details) Let $S_{id}$ be the covering of $S_0$ whose fundamental group is $\phi^{-1}(id)$. We may identify $C_\infty(S_{id})$ with $[C_\infty(S_{id})]^{H_1}$. Then we have an isomorphism
\[ T_S: [C_\infty(S_{id})]^{H_1} \to [C_\infty(S_{id})]^{H_2} \]
given by
\[ (D.4) \quad T_S(f)(x) = \sum_{g \in G} c(g)f(gx) \]
which clearly commutes with the Laplacian. If we lift a defining function on $S_0$ to $S_{id}$, and denote by $T_\partial$ the functional on the boundary of $S_{id}$ given by the formula (D.4), then $T_\partial$ intertwines the scattering operator on $S_1$ with that of $S_2$.

One way to apply this theorem is by taking $S_0$ to be a closed surface (or possibly an orbifold surface) to which one can apply the Sunada construction, and then to obtain $S_0$ from $S_{id}$ by replacing neighborhoods of points of $S_{id}$ with cusps or funnels, see [8] for a discussion. In order for the singularities of $S_0$ to smooth out in the coverings, we must have that the map $\phi$ satisfies a freeness condition about the singular points of $S_0$.

If we introduce a funnel (or cusp) at a regular point of $S_0$, this will introduce $[G : H_i]$ funnels (or cusps) on $S_i$. But if we introduce a funnel at an orbifold point $p$ of order $k$, the number of funnels on $S_i$ will be $\frac{1}{k}[G : H_i]$. Note that the freeness condition guarantees that $k$ divides $[G : H_i]$.

Using this argument, we can show:

**Theorem D.2.**

1. There exist Riemann surfaces $S_1$ and $S_2$ which are of genus 4 with one funnel which are isophasal (compare [6]).
2. There exist surfaces $S_1$ and $S_2$ of genus 3 with one funnel, which are conformally equivalent and carry metrics which have constant curvature $-1$ outside of compact sets, which are isophasal (compare [7], [8]).
3. There exist families of size $(c_1)^{k^2 \log k}$ of mutually isophasal Riemann surfaces of genus $c_2 k$ with $c_4 k$ funnels (compare [6]).

One may generalize this construction in the following way: if $\overline{S_0}$ is an orbifold surface, we may take $\phi$ to be any homomorphism
\[ \phi: \pi_1(S_0) \to G, \]
which need not satisfy the freeness condition on all the orbifold points of $\overline{S}_0$. We may then introduce funnels at points of $\overline{S}_0$ as long as we introduce funnels at all the orbifold points which do not satisfy the freeness condition.

With this idea, we can show:

**Theorem D.3.**

1. **There exist two surfaces of genus 2 with four funnels which are isophasal.**
2. **There exist two Riemann surfaces of genus 3 with three funnels which are isophasal.**
3. **There exist two surfaces of genus 0 with eight funnels which are isophasal.**
4. **There exists two Riemann surfaces of genus 0 with sixteen funnels which are isophasal.**

Details of these examples will appear elsewhere.

**References**


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