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Euclidean Majorana fermions, fermionic integrals, and relative Pfaffians

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The construction of Euclidean Majorana fermions on a two-dimensional cylindrical space–time is discussed, based on a given real-time theory. After a review of the basics of defining Euclidean fermions in the free theory, a functional integral formulation for the Euclidean theory is introduced. This functional integral connects the expectation values of the Euclidean fields with the theory of relative Pfaffians. A proof, which draws on these ideas, is given for the Feynman–Kac formula of a general interacting theory.

I. INTRODUCTION

In this article, we seek to understand the theory of Euclidean Majorana fermions in terms of a fermionic functional integral, an infinite dimensional extension of the Berezin integral for a finite dimensional Grassmann algebra. The functional integral approach reveals that expectation values in the interacting theory can be written in terms of infinite-dimensional relative Pfaffians. In Ref. 2, the analytic properties of Pfaffians of skew Hilbert–Schmidt operators on Hilbert spaces were worked out in detail, and these properties are very useful when applied to the Euclidean field theory. In particular, there exists a series of regularized Pfaffians, which make it convenient to deal with the usual divergences of field theory. We will handle Wick ordering in this way. Another useful feature of the relative Pfaffian, which is one of its defining properties, is holomorphicity. It will be necessary for our purposes to extend the definition for the relative Pfaffian given in Ref. 2, but the analytic properties are preserved under this extension.

Functional integrals for fermions are not new; they have been used in presentations of the theory of Dirac fermions (see Refs. 5 and 6, for example). The point of our approach is to use the integral as a means of explaining the connection between the interacting field theory and the relative Pfaffian. Such connections appeared in the context of supersymmetry in Refs. 7 and 8. We choose to work with Majorana fermions because these fields are more fundamental in the sense that a Dirac field can be formed from a pair of Majorana fields in a simple way. The corresponding results for the Dirac fields would involve the relative determinant, of which the relative Pfaffian was constructed as a square root.

We will study fermions on a two-dimensional cylindrical space–time, although the extension to other space–time manifolds should be routine. Euclidean fermion fields were first constructed in Ref. 9, for four-dimensional Dirac fermions. The theory of Euclidean fermions was subsequently developed for other examples in articles such as Refs. 10–14. Articles concerning applications of this theory, in which Euclidean fermions are used in an essential way, include Refs. 6, 15–17. Most of these articles dealt with Dirac fermions, although Majorana fermions were discussed in some detail in Ref. 10. For completeness, we will discuss the Euclideanization of free fermions on a cylinder in some detail, following the material of Ref. 9.

For the interacting theory, we introduce an arbitrary (possibly nonlocal) Wick-ordered interaction term which is quadratic in the fermions. This includes the possibility of a Yukawa-type coupling, which is the example studied in most of the articles mentioned above. It also includes the boson–fermion interactions of the theories studied in Refs. 7 and 8. For theories

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in which the quadratic fermion interaction term involves bosonic fields, the results of Sec. VI show that when we integrate over the fermions in such theories, the resulting bosonic measures can be expressed in terms of relative Pfaffians.

We will avoid using cutoffs explicitly in this article, for the sake of notational clarity. Cutoffs are typically introduced in interacting theories to give the interaction terms nice analytic behavior. The cutoffs are then relaxed through some regularization procedure, generally after the fermions have been integrated over. Since the purpose of this article is not to discuss such regularization procedures for specific examples, we will simplify matters by assuming that the interacting term is given by an operator with a smooth kernel. We stress that the introduction of cutoff terms would be completely straightforward, and the regularized Pfaffians of Ref. 2 are a convenient way of dealing with the regularization issues.

The layout of this article is as follows. In Sec. II, we define real-time massive self-adjoint Majorana fields on the cylinder. Then in Sec. III, we construct the corresponding free Euclidean fields. As discussed in Ref. 10, it proves necessary to double the number of degrees of freedom in defining the Euclidean theory. The Euclidean fields are not self-adjoint, because the Euclidean two-point function is not positive definite. In Sec. IV, we show explicitly how the reconstruction of the real-time theory from the Euclidean works out, for completeness and because this material will be important when we consider the interacting case. This discussion parallels the material in Sec. 4 of Ref. 9. In Sec. V, we introduce the fermionic functional integral, and show how it relates to the relative Pfaffians. We introduce the quadratic interaction in Sec. VI, and use the functional integral to show that the Euclidean expectation values are just relative Pfaffians. In Sec. VII, we use a perturbation theory argument to prove the analogous results in the real-time theory, and thus prove the Feynman-Kac formula.

II. REAL-TIME MAJORANA FIELDS

We start by defining time-zero massive Majorana fields for the space-time $M:=\mathbb{R} \times T$, where $T$ is the circle defined as $[0, 2\pi]$. We use the following Majorana representation for real-time Dirac matrices:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which is the convention of Ref. 7. Real-time fields are constructed as operator-valued solutions to the Dirac equation

$$(i\partial - m)\zeta = 0,$$

where the Dirac operator $\partial:=\gamma^0 \partial_t + \gamma^1 \partial_x$.

The real-time Fock space $\mathcal{F}$ is the Hilbert space completion of the full exterior algebra over $L^2(T)$,

$$\mathcal{F}:= \wedge (L^2(T)).$$

The real-time vacuum is the function 1, which we label by $\Omega_0$. Acting on $\mathcal{F}$, according to the standard Fock space construction, are creation and annihilation operators $b(p)^*$ and $b(p)$, for momenta $p \in T = (2\pi/l)\mathbb{Z}$. These operators satisfy anticommutation relations

$$\{b(p), b(q)\} = 0, \quad \{b(p)^*, b(q)^*\} = \delta_{pq},$$

where we use the notation $\{A, B\}$ for the anticommutator $AB + BA$.

The two component field is defined by

\[ \xi_1(x) := \frac{1}{\sqrt{l}} \sum_{p \in \mathbb{T}_l} \frac{1}{\sqrt{2\omega(p)}} [\nu(-p)b(p) + \nu(p)b(-p)]e^{-ipx}, \]

\[ \xi_2(x) := -\frac{i}{\sqrt{l}} \sum_{p \in \mathbb{T}_l} \frac{1}{\sqrt{2\omega(p)}} [\nu(p)b(p) - \nu(-p)b(-p)]e^{-ipx}, \]

(25)

where

\[ \omega(p) := \sqrt{p^2 + m^2}, \quad \nu(p) := \sqrt{\omega(p) + p}. \]

(2.6)

These fields are not operators, but rather operator-valued distributions; they become bounded operators on \( \mathcal{F} \) only when convolved with suitable test functions. For a function \( f \) on \( \mathbb{T}_l \), we have

\[ \|f\|_2^2 = \frac{1}{l} \sum_{p \in \mathbb{T}_l} |\tilde{f}(p)|^2 = \frac{1}{l} \sum_{p \in \mathbb{T}_l} \frac{\nu(p)^2 + \nu(-p)^2}{2\omega(p)} |\tilde{f}(p)|^2, \]

(2.7)

where \( \tilde{f} \) is the Fourier transform

\[ \tilde{f}(p) := \int_{\mathbb{T}_l} f(x)e^{-ipx}dx. \]

(2.8)

We thus conclude immediately from Eq. (2.5) that \( \xi_\alpha(f) \), defined by

\[ \xi_\alpha(f) := \int_M f(x)\xi_\alpha(x)dx, \]

(2.9)

with \( \alpha = 1,2 \), will be a bounded operator on \( \mathcal{F} \) if and only if \( f \in L^2(\mathbb{T}_l) \).

The real-time field is "self-adjoint" in the sense that \( \xi_\alpha(f)^* = \xi_\alpha(f) \) for \( f \) a real-valued function (in general we consider complex-valued test functions). It is simple to verify that the following relations hold:

\[ \{\xi_\alpha(x), \xi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y), \]

(2.10)

\[ \langle\xi_\alpha(x)\xi_\beta(y)\rangle_{\mathcal{F}} = \frac{1}{l} \sum_{p \in \mathbb{T}_l} \frac{1}{2\omega(p)} \begin{pmatrix} \omega + p & -im \\ im & \omega - p \end{pmatrix}_{\alpha\beta} e^{-ip(x-y)}, \]

where we have denoted \( \langle \Omega_\alpha, \cdot, \Omega_\beta \rangle_{\mathcal{F}} \) simply by \( \langle \cdot \rangle_{\mathcal{F}} \). These relations are to be understood as statements about distributions on \( \mathbb{C}^2 \otimes L^2(\mathbb{T}_l) \); they do not have any meaning for specific values of \( x \) and \( y \).

The time evolution of the real-time fields is governed by the free Hamiltonian,

\[ H_0 := \sum_{p \in \mathbb{T}_l} \omega(p)b(p)b^*(p). \]

(2.11)

That is,

\[ \xi_\alpha(t,x) := e^{itH_0}\xi_\alpha(x)e^{-itH_0}, \]

(2.12)

One can easily check that this field satisfies the Dirac equation (2.2).
III. EUCLIDEAN FIELDS

The point of this section will be to define Euclidean fields from which the real-time theory can be reconstructed. For free fields, it suffices to match the Euclidean two-point function to the Schwinger two-point function associated to the real-time theory. That is, we will find the Euclidean two-point function first, and then define fields to produce such a function. In subsequent sections, we will justify this procedure by demonstrating the reconstruction of the real-time theory in both free and interacting cases.

To obtain the Schwinger function, we start by continuing the relativistic time-zero field to imaginary times,

$\hat{a}(x) := e^{-\xi_0 H_0} \xi_\alpha(x) e^{\xi_0 H_0}$, \hspace{1cm} (3.1)

where $x = (x_0,x_1) \in M$. Define the antitime-ordering operator $\tilde{T}$ by

$\tilde{T} \xi_\alpha(x) \xi_\beta(y) := \begin{cases} 
\xi_\alpha(x) \xi_\beta(y), & \text{if } x_0 < y_0; \\
-\xi_\beta(y) \xi_\alpha(x), & \text{if } y_0 < x_0.
\end{cases}$ \hspace{1cm} (3.2)

The Schwinger two-point function is defined by

$S_{\alpha\beta}(x;y) := \langle \tilde{T} \xi_\alpha(x) \xi_\beta(y) \rangle_\mathcal{F}$, \hspace{1cm} (3.3)

for $x_0 \neq y_0$. Since the fields are operator-valued distributions, the expectation value of two fields in Eq. (3.3) results in a distribution in two variables. Writing $S$ as a function of $x$ and $y$ is a notational convenience, but we should keep in mind that $S_{\alpha\beta}(x;y)$ has no meaning for particular values of $x$ and $y$. This being the case, we can simply ignore the fact that Eq. (3.3) applies only when $x_0 \neq y_0$, by choosing the unique extension of $S$ which has no singular behavior at equal times.

Of course, this is an arbitrary choice, and since the only criterion for defining $S$ is the eventual recovery of the real-time theory from the Euclidean fields, we must allow the possibility of adding a term proportional to $\delta(x_0 - y_0)$ to the definition. In the free theory, such a term would be irrelevant. In order to define an interacting theory, however, we would need to subtract off any such term through Wick ordering. Therefore, it only really makes sense to define $S$ using the nonsingular extension.

For $x_0 < y_0$, we evaluate the expression (3.3) explicitly and obtain

$S_{\alpha\beta}(x;y) = \langle \tilde{T} \xi_\alpha(x) e^{-((y_0 - x_0)H_0) \xi_\beta(y)} \rangle_\mathcal{F}$

$= \frac{1}{i} \sum_{p_1 \in T'_i} \frac{1}{2\omega(p_1)} \left( \frac{\omega(p_1) + p_1}{im} \frac{-im}{\omega(p_1) - p_1} \right) e^{-(y_0 - x_0)\omega(p_1)e^{-ip_1(x_1 - y_1)}}$. \hspace{1cm} (3.4)

Likewise, if $y_0 < x_0$,

$S_{\alpha\beta}(x;y) = \frac{1}{i} \sum_{p_1 \in T'_i} \frac{1}{2\omega(p_1)} \left( \frac{\omega(p_1) + p_1}{im} \frac{-im}{\omega(p_1) - p_1} \right) e^{-(y_0 - x_0)\omega(p_1)e^{-ip_1(x_1 - y_1)}}$. \hspace{1cm} (3.5)

To proceed, we need to rewrite these expressions in a form which looks like a Euclidean two-point function. We can do this by introducing an integral over momentum in the time direction.

Let $\mathcal{H}^\nu(M)$ be the Sobolev space of order $\nu$ on $M$, defined as the Hilbert space of functions on $M$ with the inner product
(f,g)_\nu := \sum_{p_1 \in T_i} \int_{\mathbb{R}} \int_{M \times M} f(x) \frac{e^{-ip \cdot (x-y)}}{(p^2 + m^2)^{1/2}} g(y) dx \, dy \, dp_0, \quad (3.6)

where \( m^2 > 0 \) (different values of \( m \) give equivalent norms), and \( p \cdot x \) is the Euclidean scalar product \( p_0 x_0 + p_1 x_1 \). The space \( \mathcal{H}^0(M) \) is isomorphic to \( L^2(M) \). If \( f \) is a function in \( \mathcal{H}^{-1/2}(M) \), then the expression

\[
\sum_{p_1 \in T_i} \int_{\mathbb{R}} \int_{M} \frac{p_0^n e^{-ip \cdot y}}{p^2 + m^2} f(y) dy \, dp_0 \quad (3.7)
\]
is well-defined for \( n = 0 \) or \( 1 \). We can use contour integrals to evaluate the \( p_0 \) integral in an expression of the form (3.7), to give

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \int_{M} \frac{-ip_0^n e^{-ip \cdot y}}{p^2 + m^2} f(y) dy \, dp_0 = \frac{\omega(p_1)^n e^{ip_1 \cdot y}}{2\omega(p_1)} \int_{T_i} \int_{-\infty}^{0} e^{-\omega(p_1) y_0} f(y_0) \, dy_0 \left[ - \int_{0}^{\infty} e^{\omega(p_1) y_0} f(y_0) \, dy_0 \right] dy_1. \quad (3.8)
\]

Using Eq. (3.8) in conjunction with Eqs. (3.4) and (3.5), we obtain the following proposition. Let \( \mathcal{H} \) denote the Hilbert space

\[
\mathcal{H} := C^2 \otimes \mathcal{H}^{-1/2}(M). \quad (3.9)
\]

**Proposition III.1:** The Schwinger two-point function defined by Eq. (3.3) is a skew bilinear form on \( \mathcal{H} \otimes \mathcal{H} \), which has the following integral kernel representation:

\[
S(x,y) = \frac{1}{2\pi l} \sum_{p_1 \in T_i} \int_{\mathbb{R}} \left( \begin{array}{cc} p_1 - ip_0 & -im \\
 im & -p_1 - ip_0 \end{array} \right) \frac{e^{-ip \cdot (x-y)}}{p^2 + m^2} dp_0. \quad (3.10)
\]

The two-form \( S \) can alternatively be thought of as a skew element of \( \mathcal{H}' \otimes \mathcal{H}' \) or as a skew map from \( \mathcal{H} \) to \( \mathcal{H}' \). As a map, it can be written

\[
S = [-i\gamma_2^0 (\gamma_2 + m)]^{-1}, \quad (3.11)
\]

where the Euclidean Dirac matrices are \( \gamma_2^0 = i\gamma^0, \gamma_2^1 = \gamma^1 \), and the Euclidean Dirac operator is defined by \( \gamma_2 = \gamma_2^0 \eta_0 + \gamma_2^1 \partial_1 \).

We see immediately from Eq. (3.10) that \( S \) cannot be the two-point function of a self-adjoint Euclidean field because \( S_{\eta\eta}(x,y) \neq S_{\eta \eta}(y,x) \). According to the arguments of Ref. 10, the only way to define Euclidean fields which reproduce the Schwinger two-point function without introducing extra singularities is to double the number of degrees of freedom. That is, whereas the real-time time-zero Fock space was \( \Lambda L^2(T) \), the Euclidean Fock space will be

\[
\mathcal{F} = \Lambda (C^2 \otimes L_2(\mathbb{R} \times T_i)), \quad (3.12)
\]

where as before \( \Lambda \) denotes the Hilbert space completion of the full exterior algebra.

We take the Euclidean vacuum to be \( \Omega_0^2 := (1,1) \in \mathcal{F} \). Define creation and annihilation operators \( B_\sigma(p) \) and \( B_\sigma(p) \), with \( \sigma \in \{1,2\} \), \( p \in M' = \mathbb{R} \times (2\pi/l)\mathbb{Z} \), which satisfy

\[
\{B_\sigma(p), B_\gamma(q)^*\} = \delta_{\sigma\gamma} \delta(p_0 - q_0) \delta_{p_1 q_1}, \quad (3.13)
\]

The index $\sigma$ indicates on which component of $\mathcal{B}$ the operators act, under a fixed decomposition of the $\mathbb{C}^2$ in the definition (3.12) into $\mathbb{C} \oplus \mathbb{C}$.

We are now ready to define the Euclidean field, which will be a distribution on $\mathcal{X}$ with values in the space of bounded operators on $\mathcal{B}$. The test function space $\mathcal{X}$ is the largest possible because of Proposition III.1. The most general possibility for the fields is

$$
\xi_\alpha(x) = \frac{1}{2\pi l} \sum_{p_1 \in T_1} \int_{\mathbb{R}} \frac{1}{(p^2 + m^2)^{1/2}} \sum_{\sigma = 0, 1} [V_{\alpha\sigma}(-p) B_\sigma(p)^* + W_{\alpha\sigma}(p) B_\sigma(-p)] e^{-ip \cdot x} dp_0,
$$

(3.14)

with $V$ and $W$ two arbitrary $2 \times 2$ matrix valued functions of $p$. We emphasize that the first index on these matrices refers to the component of $\mathcal{X}$, and the second refers to the component of $\mathcal{B}$.

The conditions on $W$ and $V$ are the following. In order to reproduce the correct two-point function, we require

$$
W(p) V^T(p) = \begin{pmatrix} p_1 - ip_0 & -im \\ im & -p_1 - ip_0 \end{pmatrix}.
$$

(3.15)

For the fields to be as regular as possible (which means that none of the field components or their adjoints is more singular than another), we require that there exists a constant $C$ such that

$$
| V_{\alpha\beta}(p) | < C(p^2 + m^2)^{1/4}, \quad | W_{\alpha\beta}(p) | < C(p^2 + m^2)^{1/4},
$$

(3.16)

for $\alpha, \beta = 1, 2$. For convenience, in order that the expectation values of $\xi_\alpha(x) \xi_\beta(y)$ and $\xi_\beta(y) \xi_\alpha(x)^*$ be equal, we also specify that

$$
\det V(p) = \det W(p).
$$

(3.17)

It is easy to find matrices which satisfy Eqs. (3.15)-(3.17). For example, we could set

$$
V_{aa}(p) = (p^2 + m^2)^{1/4} \delta_{aa},
$$

(3.18)

$$
W_{aa}(p) = (p^2 + m^2)^{-1/4} \begin{pmatrix} p_1 - ip_0 & -im \\ im & -p_1 - ip_0 \end{pmatrix}_{aa}.
$$

Nothing we do will depend on a particular choice of these matrices.

We denote the convolutions of Euclidean operators with test functions by

$$
\xi_\alpha(f) := \int_M f(x) \xi_\alpha(x) dx, \quad \xi^*_\alpha(f) := \int_M f(x) \xi^*_\alpha(x) dx,
$$

(3.19)

for $f \in \mathcal{H}^{-1/2}(M)$, or more conveniently by

$$
\xi(f) := \sum_{\alpha = 1, 2} \int_M f_\alpha(x) \xi_\alpha(x) dx,
$$

$$
\xi^*(f) := \sum_{\alpha = 1, 2} \int_M f_\alpha(x) \xi^*_\alpha(x) dx,
$$

(3.20)

for $f \in \mathcal{H}$. Note that in this notation we have $\xi(f)^* = \xi^*(\bar{f})$. 

Proposition III.2: The conditions (3.15) and (3.17) imply the following expectation values and anticommutation relations:

\[ \langle \xi_a(x) \xi_b(y) \rangle^g = S_{ab}(x,y), \]
\[ \langle \xi_a(x) \xi_b^*(y) \rangle^g = \frac{\delta_{ab}}{2\pi i} \sum_{p_i \in \mathcal{T}_l} \int \frac{e^{-ip \cdot (x-y)}}{(p^2 + m^2)^{1/2}} dp_0, \]
\[ \{ \xi_a(x), \xi_b^*(y) \} = 0, \quad \{ \xi_a(x) \xi_b^*(y) \} = \frac{\delta_{ab}}{\pi l} \sum_{p_i \in \mathcal{T}_l} \int \frac{e^{-ip \cdot (x-y)}}{(p^2 + m^2)^{1/2}} dp_0, \]

where \( \langle \cdot \rangle^g := \langle \Omega_0^g, \cdot \Omega_0^g \rangle \) denotes the Euclidean vacuum expectation value. Furthermore, Eq. (3.16) implies that the operators \( \xi(f) \) and \( \xi^*(f) \) are bounded on \( \mathcal{F} \) if and only if \( f \in \mathcal{K} \).

**Proof:** The relations (3.21) are proven by simple explicit computation. The second statement holds because Eq. (3.16) implies that the coefficients of the \( \mathcal{B} \)'s and \( \mathcal{B}^* \)'s in the definition of \( \xi \) are uniformly bounded by \( (p^2 + m^2)^{-1/4} \), and Eq. (3.15) implies that one cannot improve on these bounds.

**IV. RECONSTRUCTION OF \( \mathcal{F} \)**

We must now justify our construction of the Euclidean fields by showing that we can use them to reconstruct the real-time Fock space \( \mathcal{F} \) and the real-time field. Only a certain subspace of the Euclidean Fock space will be relevant for the reconstruction (which is why there is so much latitude in the definition of Euclidean fields).

Define the positive time subspace \( \mathcal{F}_+ \subset \mathcal{F} \) to be the completion of the set of vectors generated from the vacuum \( \Omega_0^g \) by combinations of operators of the form \( \xi(f) \), with \( f \in \mathcal{K} \) such that \( f \) has support only for positive times \( x_0 > 0 \). Note that this definition does not involve the adjoint field \( \xi^* \).

Time reflection is the unitary involution \( \Theta \) on \( \mathcal{F} \) which is defined on a dense set of vectors by

\[ \Theta \xi(f_1) \cdots \xi(f_n) \Omega_0^g = \xi^* (\theta f_1) \cdots \xi^* (\theta f_n) \Omega_0^g, \]

where \( n = 0, 1, 2, \ldots \), \( f_k \in \mathcal{K} \), and \( \theta f(x_0, x_1) = f(-x_0, x_1) \).

We will prove reflection positivity and obtain the reconstruction of \( \mathcal{F} \) from \( \mathcal{F}_+ \) by defining a bounded linear map \( W: \mathcal{F}_+ \rightarrow \mathcal{F} \) such that

\[ \langle WX, WY \rangle_\mathcal{F} = \langle \Theta X, Y \rangle_\mathcal{F}, \]

for all \( X, Y \in \mathcal{F}_+ \). Because of this, the real-time Fock space is isomorphic to \( \mathcal{F}_+ / \ker W \). The action of time-zero real-time fields can be recovered from certain limits of Euclidean fields acting on \( \mathcal{F}_+ / \ker W \), and the real-time Hamiltonian which determines the time evolution of these fields is obtained from the Euclidean time translation operator. The rest of this section will be devoted to establishing these statements.

The map \( W: \mathcal{F}_+ \rightarrow \mathcal{F} \) can be most simply defined by its action on the vectors which consist of a Wick-ordered monomial acting on the vacuum state. This set is dense, since any product of operators can be written as a sum of Wick-ordered products. Let

\[ W: \xi(f_1) \cdots \xi(f_n) : \Omega_0^g := \hat{\xi}(f_1) \cdots \hat{\xi}(f_n) : \Omega_0, \]

where \( n = 0, 1, 2, \ldots \), and each \( f_k(x) \in \mathcal{K} \) has support only for \( x_0 > 0 \). The field \( \hat{\xi} \) is the real-time field continued to imaginary time, defined by Eq. (3.1).
Proposition IV.1: $W$, as specified by Eq. (4.3), is well-defined as a map from $\mathcal{F} \to \mathcal{F}$.

Proof: We need to check that the expression on the right-hand side of Eq. (4.3) lies in $\mathcal{F}$. The effect of the Wick ordering, for operators acting on the vacuum, is simply to eliminate any terms in the product of $\xi$'s which involves the $b$ operators. The ordering would move these $b$'s all the way to the right, and $b(p_1)\Omega_0=0$. The continuation of $b(p_1)^*$ to imaginary times is given by

$$e^{-\alpha(p_1)}b(p_1)^*e^{\alpha(p_1)} = b(p_1)^*e^{-\alpha(p_1)}.$$  

(4.4)

For a function $f(y) \in \mathcal{H}$ with support at positive times, define

$$\tilde{f}^W(p_1) = \int_M e^{-i\alpha(p_1)\Omega_0} f(y) dy,$$  

(4.5)

and let $f^W$ be the inverse Fourier transform of $\tilde{f}^W$. Then according to Eqs. (4.4) and (3.1), we can write

$$\xi(f_1) \cdot \cdots \cdot \xi(f_n) : \Omega_0 = \xi(f_1^W) \cdot \cdots \cdot \xi(f_n^W) : \Omega_0.$$  

(4.6)

It remains to show that $f^W \in \mathbb{C}^2 \otimes L_2(T_i)$. Using the fact that $f_\alpha \in \mathcal{H}^{-1/2}(M)$, with support at positive times, we can do a contour integration to evaluate

$$\int_M d\alpha \int_M \frac{|\tilde{f}_\alpha(p)|^2}{p^2+m^2} \leq \int_M \int_M \frac{e^{i\cdot(x-y)}}{p^2+m^2} f_\alpha(x) f_\alpha(y) dx \, dy \, dp_0$$

$$= \frac{\pi}{(p_1^2+m_1^2)^{1/2}} \int_M \int_M \frac{e^{i\alpha(x_1-y_1)-\alpha(x_0+y_0)} f_\alpha(x) f_\alpha(y)}{p^2+m^2} dx \, dy$$

$$= \frac{\pi}{(p_1^2+m_1^2)^{1/2}} |\tilde{f}_\alpha(p_1)|^2.$$  

(4.7)

From this we see that

$$|\tilde{f}_\alpha(p_1)|^2 \leq \frac{1}{\pi} \int \frac{|\tilde{f}_\alpha(p)|^2}{(p^2+m^2)^{1/2}} dp_0,$$  

(4.8)

which implies directly that

$$\|f^W\|_{L^2(T_i)} \leq 2\|f\|_{\mathcal{H}}.$$  

(4.9)

Proposition IV.2: For $X, Y \in \mathcal{F}_+$,

$$\langle W X, W Y \rangle_{\mathcal{F}} = \langle \Theta X, Y \rangle_{\mathcal{F}}.$$  

(4.10)

Proof: It suffices to prove this relation for vectors of the form

$$X = \xi(f_1) \cdot \cdots \cdot \xi(f_n) : \Omega_0^E, \quad Y = \xi(g_1) \cdot \cdots \cdot \xi(g_n) : \Omega_0^E,$$  

(4.11)

where $n=0,1,2,\ldots$, and all $f_k, g_k \in \mathcal{H}$ are supported only at positive times. Because $W$ is bounded and linear, the result will then extend to all of $\mathcal{F}_+$.

We write out explicitly

\begin{equation}
\langle \Theta X, Y \rangle_g = \langle \Omega_0^E : \xi(\partial f_n) \cdots \xi(\partial f_1) \cdots \xi(g_1) \cdots \xi(g_n) : \Omega_0^E \rangle_g .
\end{equation}

Using Wick's theorem to expand this expression in terms of two-point functions gives

\begin{equation}
\langle \Theta X, Y \rangle_g = \sum_{\pi \in S_n} (-1)^{\varepsilon(\pi)} \prod_{1 \leq k \leq n} \langle \xi(\partial f_k) \xi(g_{\pi(k)}) \rangle_g ,
\end{equation}

where \( \varepsilon(\pi) \) denotes the parity of the permutation \( \pi \). Now, the Euclidean fields were constructed to have a two-point function given by the Schwinger function. A special case of this is

\begin{equation}
\langle \xi(\partial f) \xi(g) \rangle_g = \langle \hat{\xi}(\partial f) \hat{\xi}(g) \rangle_{\mathcal{F}} ,
\end{equation}

where no time-ordering appears on the right-hand side because \( g \) has support only for positive times, and \( \partial f \) only for negative. We can substitute this into Eq. (4.13) and apply Wick's theorem in reverse to get

\begin{equation}
\langle \Theta X, Y \rangle_g = \langle \Omega_0^E : \xi(\partial f_n) \cdots \xi(\partial f_1) \cdots \xi(g_1) \cdots \xi(g_n) : \Omega_0^E \rangle_{\mathcal{F}} .
\end{equation}

Finally, we have \( \hat{\xi}(\partial f)^* = \hat{\xi}(f) \), since the field is self-adjoint, and this completes the proof. \( \square \)

**Proposition IV.3:** For any vector \( X \in \mathcal{F} \), we can define a family of vectors \( Y_s \in \mathcal{F}_+ \) for \( s > 0 \) such that

\begin{equation}
\lim_{s \to 0} W Y_s = X .
\end{equation}

**Proof:** As before, it suffices to consider

\begin{equation}
X := \xi(g_1) \cdots \xi(g_n) ,
\end{equation}

where \( n = 0,1,2,..., \) and \( g_k \in C^2 \otimes L^2(T) \), because such vectors are dense in \( \mathcal{F} \). Given \( h \in L^2(T) \), we can define a function \( h_s(x) \in \mathcal{F}_0^{S} - 1/2(M) \) by

\begin{equation}
h_s(x) := \frac{1}{s} \chi_{0,s}(x_0) h(x_1) ,
\end{equation}

where \( \chi_{0,s} \) is the characteristic function of the interval \([0,s]\). We can see directly from Eq. (5.5) that

\begin{equation}
\tilde{h}_s(p_1) = \frac{1}{\omega(p_1)s} (1 - e^{-\omega(p_1)s}) \tilde{h}(p_1) .
\end{equation}

The function \( h_s \) does not have a limit in \( \mathcal{F}_0^{S} - 1/2(M) \) as \( s \to 0 \). However, in \( L^2(T) \) it follows directly from Eq. (4.19) that:

\begin{equation}
h_s^W \to h ,
\end{equation}

as \( s \to 0 \).

We claim that \( Y_s \), defined by

\begin{equation}
Y_s := \frac{1}{s^n} \xi(\chi_{0,s}) \cdots \xi(\chi_{0,s}) : \Omega_0^E ,
\end{equation}

will satisfy Eq. (4.16). This is a direct consequence of Eq. (4.20). An immediate consequence of Proposition IV.3 is the following.
Proposition IV.4: $W(\mathcal{E}_+)$ is dense in $\mathcal{F}$.

At first glance, there appears to be an inconsistency in the limiting procedure specified in Proposition IV.3. The Euclidean fields have a vanishing anticommutator whereas the real-time fields do not, but we are claiming to reconstruct real-time fields through a limit of Euclidean fields. It is natural to wonder how the singularity in the real-time anticommutator appears in the limit. We will now explain this with an example.

Suppose we want to obtain the vector

$$\xi(f)\xi(g)\Omega_0 \in \mathcal{F},$$

$f, g \in \mathcal{H}$, through the limiting procedure. To apply Proposition IV.3, we first must rewrite this in terms of Wick-ordered products,

$$\xi(f)\xi(g)\Omega_0 = \xi(f)\xi(g)\Omega_0 + \langle \xi(f)\xi(g) \rangle_{\mathcal{F}} \Omega_0.$$  \hfill (4.23)

For the first term on the right-hand side, we can apply Proposition IV.3 directly, to get

$$\xi(f)\xi(g)\Omega_0 = \lim_{s \to 0} \frac{1}{\mathcal{E}} W^S_{\xi}(\chi_{0,s}f)\xi(\chi_{0,s}g)\Omega_0^E.$$  \hfill (4.24)

For the second term in Eq. (4.23), we have

$$\langle \xi(f)\xi(g) \rangle_{\mathcal{F}} = \langle \xi(f)\xi(g) \rangle_{\mathcal{F}} \Omega_0.$$  \hfill (4.25)

By Proposition IV.3, we have

$$\langle \xi(f)\xi(g) \rangle_{\mathcal{F}} = \lim_{s_1,s_2 \to 0} \frac{1}{\mathcal{E}} \langle W^S_{\xi}(\chi_{0,s_1}f)\xi(\chi_{0,s_2}g)\Omega_0^E \rangle_{\mathcal{F}}.$$  \hfill (4.26)

and thus, by Proposition IV.2,

$$\langle \xi(f)\xi(g) \rangle_{\mathcal{F}} = \lim_{s_1,s_2 \to 0} \frac{1}{\mathcal{E}} \langle \xi(\chi_{-s_1,0}f)\xi(\chi_{0,s_2}g) \rangle_{\mathcal{F}}.$$  \hfill (4.27)

Note that the interval of the first characteristic function has been inverted.

If we return to Eq. (4.23) and consider now the anticommutator of the two fields, we can see what happens in the limit. The terms of the form (4.24) vanish when we take the anticommutator, because the anticommutator of two Euclidean fields is zero. However, we cannot apply this reasoning to Eq. (4.27) because of the change in the characteristic function. We can easily check that

$$\{\xi(f),\xi(g)\} = \lim_{s_1,s_2 \to 0} \frac{1}{\mathcal{E}} \langle \xi(\chi_{-s_1,0}f)\xi(\chi_{0,s_2}g) - \xi(\chi_{-s_2,0}g)\xi(\chi_{0,s_1}f) \rangle_{\mathcal{F}},$$  \hfill (4.28)

so the limiting procedure is consistent.

The final step in our construction is the reconstruction of the free Hamiltonian from Euclidean time translation.

Proposition IV.5: For $t > 0$ and $X \in \mathcal{E}_+$,
where $T'$ is Euclidean time translation and $H_0$ is the free real-time Hamiltonian.

**Proof:** As usual, by the linearity and boundedness of $W$, it suffices to let

$$X = i \xi(f_1) \cdots \xi(f_n) :\Omega_0^E :,$$

(4.30)

where $n = 0,1,2,\ldots$, and each $f_k \in \mathcal{K}$ has support only at positive times. Applying time translation to $X$ gives

$$T'X := i \xi(f'_1) \cdots \xi(f'_n) :\Omega_0^E :,$$

(4.31)

where $f'_k(x) := f_k(x_0 - t, x_1)$. We then have

$$WT'X = \prod_{1 \leq k \leq n} (e^{-itH_0}\xi(f_k)e^{itH_0}) :\Omega_0^E :.$$

(4.32)

In view of Eqs. (4.4) and (4.6), this can be written as

$$WT'X = \sum_{q_1, \ldots, q_n \in T_i} e^{-s[q_1] + \ldots + s[q_n]} \left( \prod_{1 \leq k \leq n} C_\alpha(q_k) \tilde{f}_k^{\mathcal{W}}(q_k) b(q_k) \right) :\Omega_0^E :.$$

(4.33)

where $C_\alpha(p)$ is the coefficient of $b(p)^*$ in the definition of $\xi_\alpha$. Using the definition of $H_0$, this reduces to Eq. (4.29).

**V. THE FUNCTIONAL INTEGRAL AND RELATIVE PFAFFIANS**

Let $\mathcal{G}$ be the infinite-dimensional algebra generated by polynomials in the operators $\xi(f)$ for $f \in \mathcal{K}$. The vacuum expectation $\langle \cdot \rangle_{\mathcal{G}}$ restricts to a linear functional on $\mathcal{G}$ which we can represent as a functional integral. The fermionic covariance $S$ has an inverse

$$S^{-1}(x,y) = \frac{1}{2\pi i} \sum_{P \in T_i} \int_{\mathbb{R}} \left( \begin{array}{cc} P_1 + iP_0 & -im \\ -im & -P_1 + iP_0 \end{array} \right) e^{-ip \cdot (x-y)} dp_0 = -i\gamma_0^R(\partial_E + m),$$

(5.1)

where $\partial = \gamma_0^R \partial_\mu$. We can think of $S^{-1}$ as a quadratic form on $\mathcal{K}$. The functional integral we need to define is the Gaussian integral with covariance $S$, which schematically looks like

$$d\mu_x = \exp \left[ -\frac{i}{2} \int \xi_1^R(\partial_E + m) \xi_2 \right] d\xi_1 d\xi_2.$$

(5.2)

Equation (5.2) is not a definition because there is no Berezin integral $d\xi_1 d\xi_2$ in infinite dimensions. However, because $\mathcal{G}$ contains only polynomials, the infinite-dimensional Gaussian measure is relatively simple to obtain in this case. The integral (5.2) is defined in finite dimensions in Ref. 1, and for polynomials, taking the limit of this integral as the number of dimensions goes to infinity is straightforward. The result of this procedure is the following definition, where we adopt the notation

$$\xi(x)S(x,y)\xi(y) := \sum_{\alpha, \beta = 1, 2} \xi_\alpha(x) S_{\alpha\beta}(x,y) \xi_\beta(y),$$

(5.3)

which we will use in the remainder of the article.

**Definition V.1:** Let $\int_\mathcal{G} d\mu_x = 1$, and for $n = 1, 2, 3, \ldots, f_1, \ldots, f_n \in \mathcal{K}$, let
\[ \int_{\mathcal{G}} \xi(f_1) \cdots \xi(f_n) d\mu_\mathcal{G} := \text{Pf} \ A, \tag{5.4} \]

where \( A \) is the \( 2m \)-dimensional skew-symmetric matrix whose entries are defined by

\[ A_{ij} := \int_{\mathcal{M} \times \mathcal{M}} f_i(x)S(x;y)f_j(y)dx \, dy. \tag{5.5} \]

**Proposition V.2:** For \( \psi \in \mathcal{G} \), \n
\[ \int_{\mathcal{G}} \psi d\mu_\mathcal{G} = \langle \psi \rangle_\mathcal{G}. \tag{5.6} \]

**Proof:** This is a simple explicit computation. \( \square \)

The next issue is to find the class of functions \( R(x;y) \in \mathcal{H} \otimes \mathcal{H} \) for which the expression

\[ \exp \left( \frac{1}{2} \int_{\mathcal{M} \times \mathcal{M}} \xi(x)R(x;y)\xi(y)dx \, dy \right) \tag{5.7} \]

is integrable and for which the integral can be expressed as a relative Pfaffian. This is the type of term which will appear as a quadratic interaction in field theory. It turns out that the argument of the exponential will need to be Wick-ordered in our field theory example, but for simplicity we will deal with the nonordered case first. By integrable, we mean that the integral will exist as a limit of integrals over polynomials. Our goal is to relate these limits to relative Pfaffians. By Proposition V.2 and by continuity, we can then express the Euclidean expectation values as relative Pfaffians. To start with, assume that \( R \) is given by a finite-dimensional sum of products of functions of \( x \) and \( y \), so that Eq. (5.7) becomes a polynomial. In this case, we can choose an appropriate orthonormal basis \( \{ h_k \}, k > 1 \), for \( \mathcal{H} \) such that

\[ R_n(x;y) = \sum_{1 \leq j,k \leq n} R_{jk}h_j(x)h_k(y), \tag{5.8} \]

for some \( n \), where \( R_{jk} = -R_{kj} \). The integral of Eq. (5.7) with respect to \( d\mu_\mathcal{G} \), reduces to an \( n \)-dimensional Berezin integral. The formula for Berezin integration of a Gaussian is

\[ \int e^{(1/2)\eta^T A \eta} d\eta = \text{Pf} A, \tag{5.9} \]

up to a sign depending on orientation. The integral of Eq. (5.7) thus gives

\[ \int_{\mathcal{G}} \exp \left[ \frac{1}{2} \int_{\mathcal{M} \times \mathcal{M}} \xi(x)R_n(x;y)\xi(y)dx \, dy \right] d\mu_\mathcal{G} = \frac{\text{Pf}(S_n^{-1} - R_n)}{\text{Pf}(S_n^{-1})}, \tag{5.10} \]

where \( R_n \) denotes the matrix appearing in Eq. (5.8), and the entries of \( S_n^{-1} \) are the corresponding matrix elements for \( S^{-1} \),

\[ (S_n^{-1})_{jk} := \int_{\mathcal{M} \times \mathcal{M}} \overline{h_j(x)}S^{-1}(x;y)h_k(y)dx \, dy. \tag{5.11} \]

The denominator on the right-hand side of Eq. (5.10) appears because of the normalization of the total integral of \( d\mu_\mathcal{G} \) to 1.

The right-hand side of Eq. (5.10) is in fact the definition of the relative Pfaffian in finite dimensions. If we consider the matrices $R_n$ to be approximations to an infinite-dimensional operator, we need to know when the limit $n \to \infty$ can be taken. In Ref. 2, the relative Pfaffian $\text{Pf}(A, B)$ was defined for $A, B$ skew Hilbert–Schmidt operators on some Hilbert space $\mathcal{H}$ with a real structure, by a formula

$$\text{Pf}(A, B) := \lim_{n \to \infty} \frac{\text{Pf}(A_n^{-1} - B_n)}{\text{Pf}(A_n^{-1})},$$

where $A_n^{-1}$ and $B_n$ denote the matrices of the restrictions of the operators to the subspace of $\mathcal{H}$ spanned by the first $n$ vectors in some orthonormal basis, and the Pfaffian on the right-hand side is the finite-dimensional Pfaffian of a skew-symmetric matrix. This limit was shown to exist independently of the basis, to extend continuously to $A$ not invertible, and to be symmetric in $A$ and $B$. In addition, the relative Pfaffian has the following properties: (i) $\text{Pf}(A, B)^2 = \text{det}(I - AB)$; (ii) For $V$ bounded and invertible,

$$\text{Pf}(V^{-1} A (V^{-1})^T, V^T B V) = \text{Pf}(A, B).$$

(iii) For $z, w \in \mathbb{C}$, $\text{Pf}(z A, w B)$ is an entire function of $z$ and $w$.

This definition of the relative Pfaffian must be extended to apply to the present situation. The covariance $S$ is not Hilbert–Schmidt, nor in any other Schatten ideal, because of the infinite volume of $M$. Furthermore, for the interacting theories we will be dealing with, $R$ will not be Hilbert–Schmidt either. We will deal with these issues after first dealing with the relatively minor fact that $R$ and $S$ do not act on the same Hilbert space.

Let $\mathcal{K} = \mathcal{H} \oplus \mathcal{K}'$. This is a Hilbert space with a real structure given by the conjugate linear dual map from $\mathcal{K} \to \mathcal{K}'$ and vice versa. We can write operators on $\mathcal{K}$ as matrices with respect to the decomposition $\mathcal{K} \oplus \mathcal{K}'$. In this sense, given operators $A: \mathcal{K} \to \mathcal{K}'$ and $B: \mathcal{K}' \to \mathcal{K}$, we can define

$$T_A := \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad T_B := \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}. \quad (5.14)$$

Because we have chosen the real structure on $\mathcal{K}$ to be given by the dual map, the notions of skewness for $A$ or $B$ coincide with those for $T_A$ or $T_B$.

**Definition V.3:** For skew Hilbert–Schmidt operators $A: \mathcal{K} \to \mathcal{K}'$ and $B: \mathcal{K}' \to \mathcal{K}$, let

$$\text{Pf}(A, B) := \text{Pf}(T_A, T_B), \quad (5.15)$$

where the right-hand side is the definition in Ref. 2.

To define the relative Pfaffian for the operators we need to deal with, it is necessary to extend Eq. (5.13) to the case of $V$ unbounded. That is, let $V$ be a bounded, invertible operator on the Schwarz space $\mathcal{C}^2 \oplus \mathcal{H}(M)$ (the space of smooth functions of rapid decrease). $V$ is then considered to act as an unbounded operator on $\mathcal{K}$.

**Definition V.4:** Suppose that $V^{-1} S (V^{-1})^T$ extends to a Hilbert–Schmidt operator $\mathcal{K} \to \mathcal{K}'$, and that $V^T R V$ extends to a Hilbert–Schmidt operator $\mathcal{K}' \to \mathcal{K}$. Then we define

$$\text{Pf}(S, R) := \text{Pf}(V^{-1} S (V^{-1})^T, V^T R V). \quad (5.16)$$

It does not follow immediately from Eq. (5.13) that this definition is independent of the choice of $V$. An easy way to prove this is by means of the following expansion, which we will also find useful in Sec. VII.
Proposition V.5: Suppose that $A$ and $B$ are operators as above, and that the trace norm $\|AB\|_1 < 1$. Then we have the absolutely convergent expansion

$$\text{Pf}(A,B) = \exp \left[ - \sum_{1 \leq m < \infty} \frac{1}{m} \text{Tr}(AB)^m \right],$$

where we use the capital $\text{Tr}$ to denote an infinite-dimensional trace.

Proof: In Ref. 18, a similar expansion is used to define the Fredholm determinant

$$\det(I - T) = \exp \left[ - \sum_{1 \leq m < \infty} \frac{1}{m} \text{Tr} T^m \right].$$

This is absolutely convergent for $T$ a trace class operator on $\mathcal{H}$ with $\|T\|_1 < 1$. For the relative Fredholm determinant $\det(I - AB)$, we thus have

$$\det(I - AB) = \exp \left[ - \sum_{1 \leq m < \infty} \frac{1}{m} \text{Tr}(AB)^m \right].$$

The proposition then follows from the fact that $\det(I - AB) = \text{Pf}(A,B)^2$, the continuity of $\text{Pf}(A,B)$, and the fact that $\text{Pf}(A,0) = 1$.

The fact that $A$ and $B$ are skew is not used at all in this expansion. This reflects the fact that choosing a holomorphic square root of a holomorphic function is always possible in a local region. The point of the relative Pfaffian is that it is a global choice of holomorphic square root, and this can only be done for skew matrices.

For reference, we point out that the trace appearing in the expansion of Proposition V.5 has the form

$$\text{Tr}(AB)^m = \int_{\mathcal{H}^m} \text{tr} \left[ A(x_1 y_1) B(y_1 x_2) \cdots A(x_m y_m) B(y_m x_1) \right] dx_1 dy_1 \cdots dx_m dy_m,$$

where $\text{tr}$ denotes a trace of two-dimensional matrices.

Proposition V.6: Definition V.4 is independent of the choice of $V$.

Proof: For $\|SR\|_1 < 1$, the result follows immediately from Proposition V.5, by the cyclicity of the trace. For a general $R$, this implies

$$\text{Pf} \left( V_1^{-1} S (V_1^{-1})^T, V_1^T z R V_1 \right) = \text{Pf} \left( V_2^{-1} S (V_2^{-1})^T, V_2^T z R V_2 \right),$$

for $z \in \mathbb{C}$, $|z| < \|SR\|_1$, and for any suitable $V_1$ and $V_2$. Since both sides are entire functions of $z$, the fact that they are equal for $|z| < \|SR\|_1$ implies that they are equal everywhere.

As we mentioned above, Wick ordering of the interaction term will be necessary in our field theory example. This is because the operator $R(x,y)$ coming from a real-time interaction will contain a delta function $\delta(x_0 - y_0)$ in the time variables (see next section). This singularity implies that $SR$ is not trace class. Wick ordering is a regularization procedure which conveniently takes care of this problem. In the physics language, the problem is that the “tadpole” diagrams are divergent, and Wick ordering amounts to subtracting off these diagrams.

This regularization can be formulated in terms of relative Pfaffians, because of the regularized Pfaffians of Ref. 2. First of all, note that Wick ordering is a well-defined operation on the algebra $\mathcal{F}$. Suppose $R$ is finite dimensional as in Eq. (5.8). Then
\[
\int_{M \times M} \xi(x)R(x;y)\xi(y):dx
dy = \int_{M \times M} \xi(x)R(x;y)\xi(y):dx
dy - \sum_{1 \leq j,k \leq n} R_{jk}\langle \xi(h_j)\xi(h_k) \rangle \delta
\]
\[
= \int_{M \times M} \xi(x)R(x;y)\xi(y):dx
dy - \text{tr}(S_nR_n).
\]

(5.22)

Now, in place of Eq. (5.10), we have
\[
\int_{\mathcal{G}} \exp \left[ - \frac{1}{2} \int_{M \times M} \xi(x)R(x;y)\xi(y):dx
dy \right] d\mu_x = \frac{\text{Pf}(S_n^{-1} - R_n)}{\text{Pf}(S_n^{-1})} e^{-(1/2)\text{tr} S_n R_n},
\]

(5.23)

which is precisely the finite-dimensional form of the first regularized Pfaffian, Pf\(_2(S_n,R_n)\). The analysis of Ref. 2 showed that the limit as \(n \to \infty\) of this expression exists for \(S, R\) in the Schatten ideal \(I_4(\mathcal{H}^n)\). This regularized Pfaffian has the same properties as listed above for the Pfaffian, with the exception that \(\text{Pf}(S,R)^2\) is the Fredholm determinant \(\det_2(I-\text{SR})\).

**Definition V.7:** Let \(V\) be a bounded, invertible operator on the Schwarz space \(C^2 \otimes \mathcal{S}(M)\) (the space of smooth functions of rapid decrease), and suppose that \(V^{-1}S(V^{-1})^T\) extends to a skew operator in \(I_4(\mathcal{H}' , \mathcal{H}'\)) , and that \(V^T RV\) extends to a skew operator in \(I_4(\mathcal{H}' , \mathcal{H}'\)) . Then we define

\[
\text{Pf}_{\text{F}}(S,R) := \text{Pf}_{\text{F}}(V^{-1}S(V^{-1})^T, V^T RV).
\]

(5.24)

**Proposition V.8:** The integral \(d\mu_x\) extends by continuity to give

\[
\int_{\mathcal{G}} \exp \left[ - \frac{1}{2} \int_{M \times M} \xi(x)R(x;y)\xi(y):dx
dy \right] d\mu_x = \text{Pf}_{\text{F}}(S,R).
\]

(5.25)

Furthermore, for \(\|SR\|_2 < 1\) we have the absolutely convergent expansion

\[
\text{Pf}_{\text{F}}(S,R) = \exp \left[ - \sum_{2 \leq m < \infty} \frac{1}{2m} \text{tr}(SR)^m \right].
\]

(5.26)

**Proof:** The statements follow immediately from Eq. (5.23) and Proposition V.5.

We now give the particular choice of operator which we will need in order to apply Definition V.7.

**Proposition V.9:** Let \(A_{\epsilon}\) be the operator on \(C^2 \otimes \mathcal{S}(M)\) given by the integral kernel

\[
A_{\epsilon}(x;y) = I_2 \otimes \left( (x_0^2 + 1)^{1/2} \sum_{p_1 \in T_1} \int_{\mathbb{R}} \frac{(p_1^2 + m^2)^{\epsilon}}{|p_0|^{\epsilon}} e^{-ip \cdot (x-y)} dp_0 \right)
\]

(5.27)

(where \(I_2\) is the identity matrix in \(C^2\)). Then for \(0 < \epsilon < 3/8\), \(A_{\epsilon}^{-1} S(A_{\epsilon}^{-1})^T\) extends to an operator in \(I_4(\mathcal{H}' , \mathcal{H}'\)) .

**Proof:** The integral kernel of the operator \(|A_{\epsilon}^{-1} S(A_{\epsilon}^{-1})^T|\) is

\[
K(x;y) = I_2 \otimes \left( (x_0^2 + 1)^{-1/2}(y_0^2 + 1)^{-1/2} \frac{1}{\pi i} \sum_{p_1 \in T_1} \int_{\mathbb{R}} \frac{|p_0|^{2\epsilon} e^{-ip \cdot (x-y)}}{(p_1^2 + m^2)^{2\epsilon}(p_0^2 + m^2)^{1/2}} dp_0 \right).
\]

(5.28)

The \(I_4\) norm is
\[ \| A_\epsilon^{-1} S (A_\epsilon^{-1})^T \|_4^4 = \int_{M^4} \text{tr} [K(w;x)K(x;y)K(y;z)K(z;w)] dw \, dx \, dy \, dz. \] (5.29)

Using
\[ \int_M \frac{e^{-(p-q) \cdot x}}{x_0^2 + 1} \, dx = \pi i \delta_{p,q} e^{-|p_0 - q_0|}, \] (5.30)
we can reduce Eq. (5.29) to
\[ \| A_\epsilon^{-1} S (A_\epsilon^{-1})^T \|_4^4 = 2 \sum_{re T_1^j} \int_R \frac{1}{(r^2 + m^2)^{8\epsilon}} \prod_{1 \leq j < 4} e^{-|k_j - k_{j+1}|} \frac{|k_j|^{2\epsilon}}{(k_j^2 + r^2 + m^2)^{1/2}} \, dk, \] (5.31)
where \( k_3 \) is identified with \( k_1 \). For \( j < 3 \), change variables to \( k'_j = k_j - k_{j+1} \). The integrals over the \( k''s \) are finite because of the exponentials. After extracting these integrals, we are left with
\[ \| A_\epsilon^{-1} S (A_\epsilon^{-1})^T \|_4^4 \leq C \sum_{re T_1^j} \int_R \frac{|k|^{8\epsilon}}{(r^2 + m^2)^{8\epsilon}(k^2 + r^2 + m^2)^{1/2}} \, dk. \] (5.32)

This will converge provided \( \epsilon > 0 \) and \( 4 - 8\epsilon > 1 \), or \( 0 < \epsilon < 3/8 \).

The relative Pfaffian minor can be defined by a simple extension of the relative Pfaffian. It appears as the result of an integral over an exponential factor of the form (5.7) times a monomial in \( \mathcal{G} \). This can be evaluated as follows. If \( S^{-1} - R \) is invertible, then we can regard the exponential (5.7) as a shift in covariance from \( S \) to \( (S^{-1} - R)^{-1} \). The change of normalization is just given by the relative Pfaffian \( \text{Pf}(S,R) \), and applying Definition V.1 for the new covariance gives
\[ \int_{\mathcal{H}} \xi(f_1) \cdots \xi(f_n) \exp \left[ \frac{1}{2} \int_{M \times M} \xi(x) R(x;y) \xi(y) \, dx \, dy \right] \, d\mu_\eta \]
\[ = \text{Pf}(S,R) \left[ \text{Pf}(S^{-1} - R)^{-1} \left[ \int_{M \times M} f_j(x) (S^{-1} - R)^{-1}(x;y) f_j(y) \, dx \, dy \right] \right], \]
(5.33)
for \( n = 1, 2, 3, \ldots \), and \( f_1, \ldots, f_n \in \mathcal{H} \). The operator \( S^{-1} - R \) will be invertible provided \( \|SR\| < 1 \), and just as before we can extend to all \( R \) by replacing \( R \) by \( zR \), \( z \in \mathbb{C} \), and using analyticity in \( z \). Thus we can define the relative Pfaffian minor by
\[ \text{Pf}(S,R;f_1,\ldots,f_n) := \text{Pf}(S,R) \left[ \text{Pf}(S^{-1} - R)^{-1} \left[ \int_{M \times M} f_j(x) (S^{-1} - R)^{-1}(x;y) f_j(y) \, dx \, dy \right] \right], \]
(5.34)
for any \( R \) for which \( \text{Pf}(S,R) \) is defined. Wick ordering affects only the change in normalization, so that
\[ \int_{\mathcal{H}} \xi(f_1) \cdots \xi(f_n) \exp \left[ \frac{1}{2} \int_{M \times M} \xi(x) R(x;y) \xi(y) \, dx \, dy \right] \, d\mu_\eta = \text{Pf}_2(S,R;f_1,\ldots,f_n), \]
(5.35)
where

\[ \text{J. Math. Phys., Vol. 34, No. 7, July 1993} \]
\[
Pf_2(S, R; f_1, \ldots, f_n) := Pf_2(S, R) \prod_{i < j < n} \int_{M \times M} f_i(x) (S^{-1} - R)^{-1}(x; y) f_j(y) \, dx \, dy.
\] (5.36)

We summarize the constructions of this section in the following theorem.

**Theorem V.10:** Let \( \mathcal{G} \) be the algebra which is the extension of \( \mathcal{B} \) by elements of the form

\[
\exp \left[ \frac{1}{2} \int_{M \times M} :\xi(x) R(x; y) \xi(y): \, dx \, dy \right],
\] (5.37)

with \( R \) a skew operator such that \( A_n^T R A_n \in I_4(\mathcal{K}', \mathcal{K}') \) for \( 0 < \varepsilon < 3/8 \), where \( A_n \) is defined by Eq. (5.27). Then the integral \( d\mu_\alpha \) on \( \mathcal{B} \) extends by continuity to \( \hat{\mathcal{G}} \), such that

\[
\int_{\mathcal{G}} \xi(f_1) \cdots \xi(f_n) \exp \left[ \frac{1}{2} \int_{M \times M} :\xi(x) R(x; y) \xi(y): \, dx \, dy \right] d\mu_\alpha = Pf_2(S, R; f_1, \ldots, f_n),
\] (5.38)

for \( f_1, \ldots, f_n \in \mathcal{K} \). Furthermore, for any \( \psi \in \hat{\mathcal{G}} \),

\[
\int_{\mathcal{G}} \psi \, d\mu_\alpha = \langle \psi \rangle_\mathcal{G}.
\] (5.39)

**Proof:** The statements follow immediately from Proposition V.8, the definition of the Pfaffian minor, and the continuity of \( \langle \cdot \rangle_\mathcal{G} \). \( \square \)

**VI. QUADRATIC INTERACTIONS**

We will now apply the functional integral to the interacting Euclidean theory. We will consider a general (possibly nonlocal) interaction term, and use the functional integral to express the expectation values in the interacting theory in terms of Pfaffians. Let \( Q \) be a skew operator on \( C^2 \otimes L^2(T_j) \) given by an integral kernel \( Q(x; y) \) on \( T_j \). As discussed in the introduction, we will assume that cutoff functions have been applied so that \( Q_{ab}(x; y) \) is a smooth function of \( x \) and \( y \) for \( a, b = 1, 2 \).

The Wick-ordered Euclidean interaction term corresponding to \( Q \) is defined for \( t > 0 \) by

\[
V_E^t := \frac{1}{2} \int_0^t \int_{T_j \times T_j} :\xi(s, x_1) Q(x_1; y_1) \xi(s, y_1): \, dx_1 \, dy_1 \, ds,
\] (6.1)

where sums over indices are suppressed as in Eq. (5.3). Define the skew operator \( R^i: \mathcal{K}' \to \mathcal{K}' \) by the integral kernel

\[
R^i(x; y) := \delta(x_0 - y_0) Q(x_1; y_1) \chi_0(t)(y_0),
\] (6.2)

and note that

\[
V_E^t = \frac{1}{2} \int_{M \times M} :\xi(x) R^i(x; y) \xi(y): \, dx \, dy.
\] (6.3)

**Proposition VI.1:** For \( \varepsilon > 1/4 \), \( A_\varepsilon^T R^i A_\varepsilon \) extends to an operator in \( I_4(\mathcal{K}', \mathcal{K}') \), for any \( t > 0 \), where \( A_\varepsilon \) is the operator defined in Eq. (5.27).

**Proof:** The integral kernel of the operator \( A_\varepsilon^T R^i A_\varepsilon \) is
\[ K(x; y) = \sum_{p_1, q_1 \in T'_f} \int_{\mathbb{R}^2} \int_{\mathcal{M} \times \mathcal{M}} \frac{e^{i\varphi \cdot (x - w)} (p^2 + m^2) \epsilon}{|p_0|^\epsilon} Q(w_1; x_1) (w_2 + 1) \chi_{0,1}(w_0) \delta(w_0 - z_0) \]

\[ \times \frac{e^{i\varphi \cdot (y - z)} (q^2 + m^2) \epsilon}{|q_0|^\epsilon} \, dw \, dz \, dq_0 \, dp_0. \] (6.4)

The $I_4$ norm is

\[ \| A^{-1}_\epsilon S(A^{-1}_\epsilon)^T \|^4_4 = \int_{\mathbb{R}^4} \text{tr} [K(w; x) * K(x; z) * K(z; y) * K(y; w)] \, dw \, dx \, dy \, dz. \] (6.5)

Because of the smoothness of $Q$ and the compactness of $T_1$, the integrals over $w_1, x_1, y_1, z_1$ and sums over $p_1, q_1$ pose no problems. Extracting these integrals, we can bound the norm by

\[ \| A^{-1}_\epsilon T'R A^{-1}_\epsilon \|^4_4 < C \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \prod_{1 < j < 4} \frac{e^{-ik_j (w_j - w_{j+1})}}{|k_j|^{2\epsilon}} (w_j^2 + 1) \chi_{0,1}(w_j) \, da \, dk. \] (6.6)

We can compute

\[ \int_{\mathbb{R}} \frac{e^{-ikw}}{|k|^{2\epsilon}} \, dk = \left( \frac{2}{\pi} \right)^{1/2} \frac{\Gamma(1 - 2\epsilon) \sin(\pi \epsilon)}{|w|^{1 - 2\epsilon}}. \] (6.7)

The resulting integrals over $w_j$ in Eq. (6.6) will be convergent for $\epsilon > 1/4$, since the worst possible singularity would be $|w|^{-1 - 2\epsilon}$.

The Pfaffian $\text{Pf}(S, R)$ is thus well-defined. Using Theorem V.10, we immediately conclude the following.

**Theorem VI.2:**

\[ \langle \exp(-V'_E) \rangle_S = \text{Pf}(S, R). \]

Furthermore, for $X, Y \in \mathcal{B}^+_+$ given by

\[ X = \xi(f_1) \cdots \xi(f_n) \Omega^E_0, \quad Y = \xi(g_1) \cdots \xi(g_m) \Omega^E_0, \] (6.8)

with $m, n = 0, 1, 2, \ldots$, we have

\[ \langle \Theta X, \exp(-V'_E) T'Y \rangle_S = \text{Pf}(S, R; \theta \tilde{f}_n, \ldots, \theta \tilde{f}_1; \tilde{g}_1, \ldots, \tilde{g}_m), \] (6.9)

where $\theta \tilde{f}(x) := f(-x_0, x_1)$, and $g'(x) := g(x_0 - l, x_1)$.

**VII. THE FEYNMAN-KAC FORMULA**

We will now study the real-time theory with a quadratic interaction term,

\[ V = \frac{1}{2} \int_{T_1 \times T_1} \xi(x) Q(x, y) \xi(y) \, dx \, dy, \] (7.1)

where $Q$ is the operator introduced in the last section, and where sums over indices are suppressed as in Eq. (5.3). In this section, we wish to prove the analog of Theorem VI.2 for the real-time interaction term (7.1).

**Theorem VII.1:** For $X, Y \in \mathcal{B}^+_+$ given by

for \(m, n = 0, 1, 2, \ldots\), we have

\[
\langle WX, e^{-i(H_0 + V)\tau} WY \rangle_{\mathcal{F}} = \text{Pf}_2(S, R) \delta \bar{f}_{n}, \ldots, \delta \bar{f}_{1}, g^m, \ldots, g^1),
\]

where \(\delta \bar{f}(x) = f(-x_0, x_1)\), and \(g'(x) = g(x_0 - t, x_1)\).

An immediate consequence of Theorem VI.2 and Theorem VII.1 is the following Feynman–Kac formula.

\[\text{Theorem VII.2: For } X, Y \text{ as in Theorem VII.1},\]

\[
\langle WX, e^{-i(H_0 + V)\tau} WY \rangle_{\mathcal{F}} = \langle \Theta X, \exp \{-V^0\tau\}, -Y \rangle_{\mathcal{F}}.
\]

**Proof of Theorem VII.1:** To illustrate the combinatorics involved, we will start by treating the vacuum to vacuum case, which is the statement

\[
\langle e^{-i(H_0 + V)\tau} \rangle_{\mathcal{F}} = \text{Pf}_2(S, R).
\]

If we let \(V_s := e^{-iH_0\tau} e^{iH_0}\), then standard perturbation expansion for the expectation in Eq. (7.5) is

\[
\langle e^{-i(H_0 + V)\tau} \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{0}^{\tau} \cdots \int_{t_n < t_{n-1} < \cdots < t_1} \langle \bar{f}(V_{t_1} \cdots V_{t_n}) \rangle_{\mathcal{F}} dt_1 \cdots dt_n.
\]

This series is absolutely convergent for \(V\) sufficiently small. We will discuss the precise criterion for convergence in a moment. By restricting the domain of integration, we can eliminate the time ordering

\[
\langle e^{-i(H_0 + V)\tau} \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} (-1)^n \int_{0}^{\tau} \cdots \int_{t_n < t_{n-1} < \cdots < t_1} \langle V e^{-i(t_1-t_2)H_0} \cdots e^{-i(t_n-t_{n-1})H_0} V \rangle_{\mathcal{F}} dt_1 \cdots dt_n
\]

\[
= \sum_{n=0}^{\infty} \sum_{\text{partitions}(n_k) \text{of} n} \prod_k I(G_{n_k}),
\]

where \(I(G_m)\) denotes the contribution from the graph which is a loop with \(m\) vertices. The effect of the Wick ordering is to remove the loops with one vertex (the "tadpole" diagrams). For a loop \(G_m\), we get factors of \((-1)^m\) to account for the overall \((-1)^n\), \((1/2)^m\) from the \(1/2\) in \(V\), and \(2^{(m-1)}(m-1)!\) ways to contract fields in the \(V^s\). This gives

\[
I(G_m) = \frac{(-1)^m}{2} (m-1)! \int_{t_n \cdots t_1 < t} \int_{t_n \cdots t_1 < t} \text{tr} [Q(x_1, y_1) W(t_2-t_1, y_1-x_2) Q(x_2, y_2)]
\]

\[
\times W(t_3-t_2, y_2-x_3) \cdots Q(x_m, y_m) W(t_m-t_1, x_1-y_m)
\]

\[
\times dx_1 \cdots dx_m dy_1 \cdots dy_m dt_1 \cdots dt_m,
\]

where \(W\) is the 2 by 2 matrix function.
Observe that

\[ S_{\alpha\beta}(x,y) = \begin{cases} \mathcal{W}_{\alpha\beta}(y_0 - x_0, x_1 - y_1), & \text{if } x_0 < y_0 \\ -\mathcal{W}_{\beta\alpha}(x_0 - y_0, y_1 - x_1), & \text{if } x_0 > y_0. \end{cases} \]  

This is simply the definition of \( S \). Since \( S \) was defined to be nonsingular at \( x_0 = y_0 \), we can ignore such points. Substituting Eq. (7.10) into Eq. (7.8) gives

\[ I(G_m) = \frac{(-1)^m}{2} (m-1)! \int_0^{t_1} \cdots \int_0^{t_m} \int_{T_{2m}}^{t_m} \text{tr}[Q(x_1, y_1)S(t_1, t_2, x_2) \cdots Q(x_m, y_m) S(t_1, t_2) \cdots S(t_m, t_1)]dx_1 dy_1 \cdots dx_m dy_m dt_1 \cdots dt_m. \]  

We can exploit the symmetry of the integrand to expand the domain of integration to \( 0 < t_1, \ldots, t_m < t \), picking up a factor of \( (1/m!) \). Recalling the definition of \( K \), the resulting expression is just

\[ I(G_m) = -\frac{(-1)^m}{2m} \text{Tr}(SR)^m. \]  

The combination \((SR)^m\), for \( m \geq 2 \), is seen to be trace class as a consequence of Proposition V.9 and Proposition VI.1.

Returning to the full perturbation series, we can perform the sum over \( n \) by simply relaxing the restriction on the sum of the \( n_k \). The result is

\[ \langle e^{-t(H_0 + V)} \rangle_{\mathcal{F}} = \exp \left[ -\sum_{2 \leq m < \infty} \frac{1}{2m} \text{Tr}(SR)^m \right], \]  

which is exactly the expansion of Proposition V.8.

Thus, using Proposition V.8, we see that the perturbation expansion is absolutely convergent provided

\[ \|SR\|_2 < 1, \]  

and we obtain Eq. (7.5) in this case. The restriction (7.14) can be relaxed as follows. Suppose \( z \in \mathbb{C} \) such that \( |z| < 1/\|SR\|_2 \). Because \( V \) and \( H_0 \) are bounded operators on \( \mathcal{F} \), the series

\[ \langle WX, e^{-t(H_0 + zV)} WY \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \langle WX, (H_0 + zV)^n WY \rangle_{\mathcal{F}} \]  

is absolutely convergent for all \( z \in \mathbb{C} \). Hence Eq. (7.15) is an entire function of \( z \). Furthermore, as a function of \( z \), \( \text{Pf}(S, zR) \) is also analytic. Because the entire functions (7.15) and \( \text{Pf}(S, zR) \) are have been proven equal for \( |z| < 1 \), they are equal for all \( z \). This proves that Eq. (7.5) holds without the restriction (7.14).
This completes the proof for the vacuum case. Consider now

$$\langle WX,e^{-i(H_0+V)WY} \rangle_{\mathcal{S}}.$$ \hfill (7.16)

In the perturbation expansion for this expression, we have two kinds of diagrams: loops, which are treated exactly as above, and terms involving two sources connected by the full propagator. The loop terms can be extracted and summed independently, as usual in field theory, to give an overall factor of $\text{Pf}(S,R)$.

To reduce the terms involving sources, we use the two new facts

$$\int_{-\infty}^{\infty} S_{ab}(t',y';s,y) T' g(s,y) ds = \int_{-\infty}^{\infty} S_{ab}(t',y';s,y) g(s-t,y) ds = W_{ab}(t-t',y-y) g^w(y),$$ \hfill (7.17)

and

$$\int_{-\infty}^{\infty} \delta g(s,x) S_{ab}(s,x;t',x') ds - \int_{-\infty}^{\infty} g(-s,x) S_{ab}(s,x;t',x') ds = \bar{g}^w(x) W_{ab}(t',x-x'),$$ \hfill (7.18)

where $g \in \mathcal{H}^{\alpha-1/2}(M)$ and $g^w \in L_2(T)$ was defined in Sec. IV. These two facts cover the possible source terms coming from $X$ and $Y$.

With some attention to combinatorics, we straightforwardly obtain

$$\langle WX,e^{-i(H_0+V)WY} \rangle_{\mathcal{S}} = \text{Pf}(S,R) \sum_{\pi \in S_{2l}} (-1)^{\epsilon(\pi)} \prod_{1 \leq k < l} \int_{M \times M} h_{\pi(2k-1)}(x)$$

$$\times D^{-1}(x,y) h_{\pi(2k)}(y) dx dy,$$ \hfill (7.19)

where $2l-m+n$, and $\{h_1,\ldots,h_{2l}\}$ is defined to be the sequence of functions

$$\{\theta f_{2l},\ldots,\theta f_1,\theta g_{m},\ldots,\theta g_1\},$$ \hfill (7.20)

and where $D$ is the full propagator, defined formally by the series

$$D = S + SRS + SRSRS + \cdots = \sum_{n=0}^{\infty} (SR)^n S.$$ \hfill (7.21)

Under the restriction (7.14), the perturbation converges absolutely, and the formal expression for $D$ converges to a well-defined, bounded operator,

$$D = (I-SR)^{-1}S = (S^{-1}-R)^{-1}.$$ \hfill (7.22)

Then, the result is just the relative Pfaffian as given in Eq. (7.19). The restriction (7.14) can be removed by an argument based on analyticity, exactly as in the vacuum case.

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